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A MULTI-REGION ECONOMIC GROWTH MODEL WITH MIGRATION, HOUSING AND REGIONAL AMENITY

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Abstract

The purpose of this paper is to examine dynamic interactions among interregional migration, economic development and environmental change on microeconomic foundation. The model describes multi-regional economic growth with heterogeneous households, and capital accumulation under assumptions of profit maximization, utility maximization, and perfect competition. First, we simulate the national economy of a three-regional economy with homogeneous population. It is demonstrated that when the production functions take on the Cobb-Douglas form, the dynamics has a unique equilibrium. Then we carry out comparative statics analysis with regard to regions' productivity and amenity levels, the propensity to save, and the population. The simulation results show that any change in any region's amenity or productivity may improve or worsen some regions with regard to some variables. We will also examine the dynamics of a two-region two-group economy.

Keywords: two-region growth model, capital accumulation, amenity, heterogeneous households
JEL classification: D11; E24; E32; E37; F22; R11; R23

1. Introduction

Firms and households take regional differentials in living conditions and productivity into account in their locational decision-making. The productivity advantages of one region may be offset to some extent by the higher wages that must be paid in a system where people are free to choose where they work and live. Higher wages are often associated with some kinds of disamenities (such as noise, pollutants, and densely populated neighborhood) and high living costs. Labor force, capital, knowledge are main inputs of economic production. Labor and capital are easily mobile between regions in industrialized economies. As capital mobility becomes high and costs associated with capital movement among regions become low, it is reasonable to assume that capital movement equalizes rate of interest between regions within a national economy. But there are different principles for analyzing temporary equilibrium conditions for labor movement in a dynamic regional framework. For example, in the theoretical literature on regional economics two principles have been proposed to analyze labor movement. One is that labor movement, if costs associated with professional or locational changes are neglected, equalizes wage rates between regions. This assumption is limited in a freely competitive national labor market if

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various regions provide different levels of amenity (such as regional cultures, climates and pollution) and have different technologies. The other one is that free movement of people equalizes utility levels, which they may obtain in different regions. Although this assumption is reasonable for analyzing issues related to movement of people between regions, the traditional approaches to consumer behavior in economic theory often make it analytically intractable to properly model migration with regional differences in life styles and wage rates. A reason is that it is difficult to examine issues related to ownership of resources and capital in a dynamic regional economy. If wage rates and life style (including saving behavior) are regionally different and people may freely move within the economy, how to calculate each individual's capital and his income in a dynamic multi-regional economy is not an easy matter. In this paper, we solve this issue by utilizing a utility function recently used by Zhang (1998). In this approach we assume that each household's preference structure over lot size, consumption and wealth is represented by the utility function of the region where it chooses to live and work at any point of time t. Zhang (2005) has explained this approach and its relations to traditional approaches to consumer behavior with endogenous saving in a recent book. As demonstrated systematically in Zhang's recent book, this approach makes it possible to solve many important (national) economic problems, such as growth problems with heterogeneous households, which are analytically intractable by the traditional approaches.

Regional dynamics of capital accumulation – not to mention innovation and environmental change - with profit-maximization and utility maximization are so difficult that few serious attempts have been made. One of reasons for the lacking of theoretical interest is that the traditional approaches to consumer behavior will result in such a complicated dynamic system that the behavior of the system becomes intractable. This paper builds a dynamic one-commodity and multiple-region trade model to examine interdependence between regional trades and national growth. We analyze trade issues within the framework of a simple international macroeconomic growth model with perfect capital mobility. This model is influenced by the neoclassical trade theory with capital accumulation. Since the publication of Oniki and Uzawa's paper on theory of trade and economic growth (Oniki and Uzawa, 1965), various trade models with endogenous capital have been proposed (e.g., Deardorff, 1973, Ruffin, 1979, Findlay, 1984, Eaton, 1987, Brecher, *et al.*, 2002, Sorger, 2003). It is well known that dynamic-optimization models with capital accumulation are associated with analytical difficulties. To avoid these difficulties, this study applies an alternative approach to consumer behavior. It will be demonstrated that the multi-region trade model with capital accumulation becomes much more analytically tractable with the new approach to consumer behavior. But different from the international trade models just referred, this paper develops a model for economic interactions among regions. Different the international trade models where people are not allowed to move spatially, this study allows people to be mobile among regions. This paper is organized as follows. Section 2 defines the multi-region model with heterogeneous households and capital accumulation. Section 3 simulates the motion of the 3-region national economy with a homogeneous population. Section 4 examines the effects of changes in different regions' amenity levels, productivity levels and residential areas, the national population, and the propensity to consume upon the national economic growth and regional economic structures. Section 5 provides the procedure for computing equilibrium of the two-region and two-group economy. Section 6 concludes the study. Appendix proves the results in Section 5.

2. The multi-region trade model with capital accumulation

In describing economic production, we follow the neoclassical trade framework. It is assumed that the regions produce a homogenous commodity (see, for instance, Ikeda and Ono, 1992). Most aspects of production sectors in our model are similar to the neo-classical one-sector growth model (Burmeister and Dobell 1970, Zhang, 2005). It is assumed that there is only one (durable) good in the national economy under consideration. Households own assets of the economy and distribute their incomes to consume and save. Production sectors or firms use capital and labor. Exchanges take place in perfectly competitive markets. Production sectors sell their product to households or to other sectors and households sell their labor and assets to production sectors. Factor markets work well; factors are inelastically supplied and the available factors are fully utilized at every moment. Saving is undertaken only by households, which implies that all earnings of firms are distributed in the form of payments to factors of production. We omit the possibility of hoarding of output in the form of non-productive inventories held by households. All savings volunteered by households are absorbed by firms. We require saving and investment to be equal at any point of time.

A person is free to choose his residential location. We assume that any person chooses the same region where he works and lives. Each region has fixed land. Land quality, climates, and environment are homogenous within each region, but they may vary among the regions. We neglect transportation cost of commodities between and within regions. As become evident later on, although it is conceptually not difficult to introduce transportation cost function and to provide balance conditions for demand and supply and for price equalization conditions with transportation cost (e.g., Takayama and Judge, 1971, Roy and Johansson, 1993), the problem will become analytically too complicated. The assumption of zero transportation cost of commodities implies price equality for the commodity among the two regions. As amenity and land are immobile, wage rates and land rent may be not equal between the two regions.

The system consists of J regions, indexed by $j = 1, \dots J$. Only one good is produced in the system. Perfect competition is assumed to prevail in good markets both within each region and between the regions, and commodities are traded without any barriers such as transport costs or tariffs. The labor markets are perfectly competitive within each region and among regions. Let prices be measured in terms of the commodity and the price of the commodity be unity. We assume that the national population is classified into Q groups, indexed by q, according to their preferences, human capital, and social status.

A group q in region j is indexed by (j, q). We introduce

$$Q^* \equiv \{(j, q) \mid j = 1, \dots, J, \;\; q = 1, \dots, Q\}.$$

Let the number of group q in region j be $N_{jq}(t)$. The aggregated labor force, $N_j(t)$, of region j is given by

$$N_j(t) = \sum_{q=1}^{Q} z_q N_{jq}(t), \;\; (j, q) \in Q^*, \tag{1}$$

where z_q are the level of human capital of group q. In this initial stage, we assume human capital to be constant. We denote wage rates and interest rate by $w_{jq}(t)$, $(j, q) \in Q^*$, and $r(t)$.

We introduce

$\bar{N}_q$ —*the fixed population of group* q;

L_j —*the fixed (residential) area of region* j;

$K(t)$ —*the total capital stocks of the economy at time* t;

$F_j(t)$ —*the output levels of region* j *'s production sector at time* t;

$K_j(t)$ *and* $N_j(t)$ —*the levels of capital stocks employed by region* j *'s production sector;*

$c_{jq}(t)$ *and* $s_{jq}(t)$ —*per capita's consumption level of commodity and saving made by per household* $(j, q) \in Q^*$;

$l_{jq}(t)$ —*the lot size of per household* $(j, q) \in Q^*$;

$R_j(t)$ —*region* j *'s land rent.*

We now describe the model.

behavior of producers

First, we describe behavior of the production sections. We use production functions to describe the physical facts of a given technology. We assume that there are only two productive factors, capital stocks, $K_j(t)$, and labor , $N_j(t)$, at point of time t. The production functions are given by

$$F_j(K_j(t), N_j(t)), \quad j = 1, \dots J.$$

Assume F_j to be neoclassical. We have

$$f_j(t) = f_j(k_j(t)), \quad f_j(t) \equiv \frac{F_j(t)}{N_j(t)}, \quad k_j(t) \equiv \frac{K_j(t)}{N_j(t)}.$$

The functions f_j have the following properties: (i) $f_j(0) = 0$; (ii) f_j are increasing, strictly concave on R^+, and C^2 on R^{++}; $f_j'(k_j) > 0$ and $f_j''(k) < 0$; and (iii) $\lim_{k_j \to 0} f_j'(k_j) = \infty$ and $\lim_{k_j \to +\infty} f_j'(k_j) = 0$.

Markets are competitive; thus labor and capital earn their marginal products, and firms earn zero profits. For any individual firm $r(t)$ and $w_{jq}(t)$ are given. Region j's production sector chooses two variables, $K_j(t)$ and $N_j(t)$, to maximize its profit. The marginal conditions are given by

$$r + \delta_{kj} = f_j^{'}(k_j), \ \ w_{jq}(t) = z_q\left(f_j(k_j) - k_j f_j^{'}(k_j)\right), \ \ (j, q) \in Q^*, \qquad (2)$$

where δ_{kj} are the depreciation rate of physical capital in region j.

behavior of consumers

Each worker may get income from land ownership, capital ownership and wages. In order to define incomes, it is necessary to determine land ownership structure. It can be seen that land properties may be distributed in multiple ways under various institutions. To simplify the model, we accept the assumption of "absent landownership" which mean that the income of land rent is spent outside the economic system. A possible case is that the land is owned by the government, people can rent the land in competitive market, and the government uses the income for military or other public purposes. As used in Zhang (1998), we may assume that each worker owns L_j / N amount of land in region j and it is impossible to sell land but it is free to rent one's own land to others. Under this assumption, the land revenue $\bar{r}$ per worker is given as follows $\bar{r}(t) = \sum_j L_j R_j / N$. Hence, each worker receives the land revenue $\bar{r}$ irrespective of his dwelling region.

Consumers make decisions on choice of consumption levels of services and commodities as well as on how much to save. There is no single purpose for people to make savings. Wealth may be accumulated for different reasons such as the capitalist spirit, old age consumption, providing education for children, power and social status. Those different reasons determine preference structures. Different from the optimal growth theory in which the utility function defined over future consumption streams is used, this study uses the approach to consumers' behavior proposed by Zhang in the early 1990s. A detailed explanation of the approach and its relations to the traditional approaches to consumer behavior are provided in Zhang (2005). Let $\bar{k}_{jq}(t)$ stand for the per capita wealth of group q in region j. A consumer q of region j obtains income

$$y_{jq}(t) = r(t)\bar{k}_{jq}(t) + w_{jq}(t), \ \ (j, q) \in Q^* \qquad (3)$$

from the interest payment, $r\bar{k}_{jq}$, and the wage payment, w_{jq}. We call y_{jq} the current income in the sense that it comes from the consumer's wage income and consumer's current earnings from ownership of wealth. The income that the consumer is using for consuming, saving, or transferring is not necessarily equal to the current income because the consumer can sell wealth to pay, for instance, the current consumption if the current income is not sufficient for buying food and touring the country. Retired people may live not only on the interest payment but also have to spend some of their wealth. The total value of wealth that a consumer of group j can sell to purchase goods and to save is equal to $\bar{k}_{jq}(t)$. Here, we assume that selling and buying wealth can be conducted instantaneously without any transaction cost. The disposable income is equal to

$$\hat{y}_{jq}(t) = y_{jq}(t) + \bar{k}_{jq}(t), \ \ (j, q) \in Q^*. \qquad (4)$$

The disposable income is used for saving and consumption.

At each point of time, a consumer distributes the total available budget among lot size, $l_{jq}(t)$, saving, $s_{jq}(t)$, consumption of goods, $c_{jq}(t)$. The budget constraints are given by

$$R_{jq}(t)l_{jq}(t) + c_{jq}(t) + s_{jq}(t) = \hat{y}_{jq}(t) = (1 + r(t))\bar{k}_{jq}(t) + w_{jq}(t). \tag{5}$$

Equation (5) means that housing, consumption and saving exhaust the consumer's disposable personal income.

We assume that utility level, $U_j(t)$, that the consumers obtain is dependent on the housing, $l_{jq}(t)$, the consumption level of commodity, $c_{jq}(t)$, and saving $s_{jq}(t)$. We specify $U_{jq}(t)$, $(j, q) \in Q^*$, as follows

$$U_{jq}(t) = \theta_{jq} l_{jq}^{\eta_{0q}}(t) c_{jq}^{\xi_{0q}}(t) s_{jq}^{\lambda_{0q}}(t), \qquad \eta_{0q}, \xi_{0q}, \lambda_{0q} > 0, \tag{6}$$

in which η_{0q}, ξ_{0q}, and λ_{0q} are the elasticities of utility of a typical person of group q with regard to lot size, commodity and saving. We call η_{0q}, ξ_{0q}, and λ_{0q} propensities to consume lot size, to consume goods, and to hold wealth (save), respectively. Consumer behavior is affected by many factors such as climates and regional cultures. In (6), θ_{jq} is the amenity level for group q in region j. Amenities in this study are considered as locational parameters or slow changing variables, such as infrastructures, regional cultures and climates, which may affect attractiveness of a location.[1]

Maximizing $U_j(t)$ subject to the budget constraints (5) yields

$$l_{jq}(t)R_j(t) = \eta_q \hat{y}_{jq}(t), \;\; c_{jq}(t) = \xi_q \hat{y}_{jq}(t), \;\; s_{jq}(t) = \lambda_q \hat{y}_{jq}(t), \;\; (j, q) \in Q^*, \tag{7}$$

in which

$$\eta_q \equiv \frac{\eta_{0q}}{\rho_q}, \;\; \xi_q \equiv \frac{\xi_{0q}}{\rho_q}, \;\; \lambda_q \equiv \frac{\lambda_{0q}}{\rho_q}, \;\; \rho_q \equiv \eta_{0q} + \xi_{0q} + \lambda_{0q}.$$

Because of the assumed Cobb-Douglas utility function, the expenditure on each variable is proportional to the disposable income. If we take on other forms of utility function, this proportional relationship between the expenditure and the disposable income will not hold. See Zhang (2005) for further discussions on behavior of the consumer when the utility function is taken on some other forms or subject to preference change. As demonstrated by numerous examples by Zhang, the utility function allows us to treat almost all the important issues in the neoclassical growth theory (without space) in a consistent manner. As shown by Zhang's recent book, if we allow the preference parameters to be related to wealth and income in certain way, then the typical household behaves the same as in the Solow model or Ramsey model.

According to the definitions of $s_{jq}(t)$, the wealth accumulation of household $(j, q) \in Q^*$ is given by

[1] For instance, Kanemoto (1980), Diamond and Tolley (1981), Blomquist et al (1988), Andersson et al (2003).

$$\dot{\bar{k}}_{jq}(t) = s_{jq}(t) - \bar{k}_{jq}(t), \ \ (j, q) \in Q^{*}. \tag{8}$$

As households are assumed to be freely mobile among the regions, the utility level of people of the same group should be equal, irrespective of in which region they live, i.e.

$$U_{jq}(t) = U_{mq}(t), \ \ j, m = 1, ... J, \ \ q = 1, ..., Q. \tag{9}$$

It should be noted that we neglect possible costs for migration. In reality, even to change a house in a small town costs. Although it is not difficult to introduce migration costs into the model, it will become far more difficult to explicitly get analytical results. In this study, instead of wage equalization (which is often used as the equilibrium mechanism of population distribution), we assume that consumers obtain the same level of utility in different regions as the equilibrium mechanism of population distribution among the regions. Although the condition of utility equalization is often used in the literature of urban equilibrium economics (see, Fujita, 1989), the assumption of utility equalization is rarely used in the literature of economic dynamics as the temporary equilibrium condition of population distribution. It is argued that this assumption is more reasonable than the assumption of wage equalization. It is well observed that wage rates vary over regions. Wage disparities may be caused by many factors, such as spatial differences in the skill composition of the workforce, in non-human endowments such (as climate and infrastructures), and in local interactions. Many empirical studies are made to study regional wage disparities in different countries (for instance, Glaeser and Maré, 2001, Duranton and Monastiriotis, 2002, Combes *et al*., 2003). As shown later on, as the utility level at each point of time in each region is related to regional wealth distribution, regional amenity levels, lands, and regional technological levels, wage disparity is “determined” by all regional factors such as resource endowments, technological levels. In general, wage rates among regions will not converge if regional differences in amenity and technology differ.

The total capital stocks employed by the production sectors is equal to the total wealth owned by the population. That is

$$K(t) = \sum_{j=1}^{J} K_j(t) \ = \sum_{j=1}^{J} \sum_{q=1}^{Q} \bar{k}_{jq}(t) N_{jq}. \tag{10}$$

The national production is equal to the national consumption and net savings. That is

$$C(t) + S(t) - K(t) + \sum_{j=1}^{J} \delta_{kj} K_j(t) = F(t), \tag{11}$$

where

$$C(t) \equiv \sum_{j=1}^{J} \sum_{q=1}^{Q} c_{jq}(t) N_{jq}, \ \ S(t) \equiv \sum_{j=1}^{J} \sum_{q=1}^{Q} s_{jq}(t) N_{jq}, \ \ F(t) \equiv \sum_{j=1}^{J} F_j(t).$$

The assumption that labor force and land are fully employed is represented by

$$\sum_{j=1}^{J} N_{jq}(t) = \bar{N}_q, \ \ \sum_{q=1}^{Q} l_{jq}(t) N_{jq}(t) = L_j, \ \ j = 1, ... J. \tag{12}$$

We have thus built the model which explains the endogenous capital accumulation and regional capital and labor distribution in the national economy in which all the markets perfectly competitive and product, capital and labor are freely mobile.

3. Simulate a 3-region model with a homogenous population

The previous section develops a general model of the national economy with multiple regions. The model is structurally general in the sense that by adding some assumptions, the model can be applied to describe national as well as international economies. For instance, if we allow all the regions have the identical conditions, then the model becomes a typical growth model of national economy. Although the behavioral mechanism of consumers of our model is different from the Solow model, the model with the identical regions is mathematically identical with the Solow model – it has a unique stable equilibrium. If we don't allow people to be interregionally mobile, then our model is mathematically identical with the Oniki-Uzawa trade model with constant saving rates. It should be noted that the Oniki-Uzawa trade model and most of its extensions are limited to two national economies. Although our model is structurally general, it is important to know whether it is analytically tractable. As argued by Zhang (2005), a main reason that the modern economic growth theory (based on the Ramsey approach to households) with capital accumulation has failed, in a consistent and comprehensive sense, to be extended to study economies with heterogeneous households, or multiple sectors, or multiple countries, or multiple regions is that any extension from the standard one-sector one-household model tends to result in a model analytically intractable. The approach based on the utility function of this study can be extended to economies with heterogeneous households and multiple sectors by Zhang. As far as dynamic analysis is concerned, the advantage of our approach over the Ramsey approach is that for similar economic issues, the dynamical dimension of our approach is lower than that by the Ramsey approach. This character makes it possible to deal with many economic issues which are much more difficult, if not impossible, to study by the Ramsey approach. It should be remarked that the recent new growth theory avoids capital accumulation and is concentrated on human capital accumulation. The omission of capital accumulation from economic theory is mainly due to the analytical difficulties, rather than on the basis of reasonable justification.

We are first concerned with a three-region model with a homogeneous population, that is, group 1 and group 2 are identical in all aspects. Hence, we omit index q. We also require $\eta + \xi + \lambda = 1$ without affecting our analysis. We specify the production functions as follows

$$F_j(t) = A_j K_j^{\alpha_j}(t) N_j^{\beta_j}(t), \ \ \alpha_j + \beta_j = 1, \ \ \alpha_j, \beta_j > 0, \quad j = 1, \cdots, J,$$

where A_j is region j's productivity. From $f_j = A_j k_j^{\alpha_j}$, where $k_j \equiv K_j / N_j$, and equations (2), we have

$$k_j = \phi_j(k_1) \equiv \left(\frac{\alpha_1 A_1 k_1^{-\beta_1} - \delta_j}{\alpha_j A_j} \right)^{-1/\beta_j}, \quad j = 2, \cdots, J.$$

By equations (2) and (4)

$$w_j = \bar{\phi}_j \equiv A_j \beta_j \phi_j^{\alpha_j}(k_1),$$

$$\hat{y}_j = \tilde{f}(k_1)\bar{k}_j + A_j \beta_j \phi_j^{\alpha_j}(k_1), \quad j = 1, \cdots, J, \tag{13}$$

where we use $\phi_1(k_1) = k_1$ and $\tilde{f}(k_1) = \alpha_1 A_1 k_1^{-\beta_1} + \delta_1$.

Substitute $c_j = \xi \hat{y}_j$ and $s_j = \lambda \hat{y}_j$ in equations (7) and $l_j = L_j / N_j$ into the utility function

$$U_j(t) = \theta_j L_j^{\eta} \xi^{\xi} \lambda^{\lambda} N_j^{-\eta} \hat{y}_j^{\xi+\lambda}, \quad j = 1, \cdots, J.$$

Inserting the above equations into equations (9), we get

$$n_j(t) = \tilde{\theta}_j \left(\frac{\hat{y}_j}{\hat{y}_1} \right)^{\overline{\eta}}, \quad j = 2, \cdots, J, \tag{14}$$

where

$$n_j(t) \equiv \frac{N_j(t)}{N_1(t)}, \quad \tilde{\theta}_j \equiv \left(\frac{\theta_j L_j^{\eta}}{\theta_1 L_1^{\eta}} \right)^{1/\eta}, \quad \overline{\eta} \equiv \frac{\xi + \lambda}{\eta}.$$

We can rewrite equation (10) as

$$\sum_{j=1}^{J} k_j(t) N_j(t) = \sum_{j=1}^{J} \bar{k}_j(t) N_j(t).$$

Insert $k_j = \phi_j(k_1)$ into the above equation

$$\bar{k}_1(t) = k_1 + \sum_{j=2}^{J} n_j \left(\phi_j(k_1) - \bar{k}_j \right). \tag{15}$$

Insert equations (14) into equation (15)

$$\bar{k}_1(t) = k_1 + \frac{1}{\hat{y}_1^{\overline{\eta}}} \sum_{j=2}^{J} \tilde{\theta}_j \hat{y}_j^{\overline{\eta}} \left(\phi_j(k_1) - \bar{k}_j \right). \tag{16}$$

Solve equation (13) for $\hat{y}_1$ with regard to $\bar{k}_1$

$$\bar{k}_1 = \frac{\hat{y}_1 - \bar{\phi}_1(k_1)}{\tilde{f}(k_1)}. \tag{17}$$

Equaling both right-hand sides of equations (16) and (17), we have

$$\hat{y}_1^{1+\overline{\eta}} - \hat{f}(k_1) \hat{y}_1^{\overline{\eta}} - \bar{f}\left(k_1, \{\bar{k}(t)\}\right) = 0, \tag{18}$$

where

$$\hat{f}(k_1) \equiv \bar{\phi}_1(k_1) + k_1 \tilde{f}(k_1) = A_1 k_1^{\alpha_1} + \delta_1 k_1 > 0,$$

$$\bar{f}\left(k_1, \{\bar{k}(t)\}\right) \equiv \tilde{f}(k_1) \sum_{j=2}^{J} \tilde{\theta}_j \hat{y}_j^{\overline{\eta}} \left(\phi_j(k_1) - \bar{k}_j \right) = \left(\alpha_1 A_1 k_1^{-\beta_1} + \delta_1 \right) \sum_{j=2}^{J} \tilde{\theta}_j \hat{y}_j^{\overline{\eta}} \left(\phi_j(k_1) - \bar{k}_j \right). \tag{19}$$

It should be noted that the functions $\hat{f}$ and $\bar{f}$ are only dependent on J variables, k_1 and $\{\bar{k}(t)\}$. This is important as it allows us to express the dynamics in a J-dimensional differential equations system.

To simulate the model, we now have to find solutions to equation (18) with $\hat{y}_1$ as the variable and $\hat{f}$ and $\tilde{f}$ as parameters. For any positive value of $\overline{\eta} = (\xi + \lambda)/\eta$, it is impossible to explicitly solve the above equation. For illustration, we specify $\bar{\eta} = 2$, which means $\xi + \lambda = 2\eta$. It should be remarked that our approach is numerically tractable with computer for any meaningful value of $\bar{\eta}$. For instance, we may use the following method. We take derivatives of equation (18) with respect to t and solve the derivative of $\hat{y}_1$ with respect to time as a function $\hat{y}_1(t)$, $k_1(t)$, $\{\bar{k}(t)\}$, $\dot{k}_1(t)$, and $\{\dot{\bar{k}}(t)\}$. As $\dot{k}_1(t)$ and $\{\dot{\bar{k}}(t)\}$ can be expressed as functions of $\hat{y}_1(t)$, $k_1(t)$ and $\{\bar{k}(t)\}$, we can express $\dot{\hat{y}}_1(t)$ as a function of $\hat{y}_1(t)$, $k_1(t)$ and $\{\bar{k}(t)\}$ as follows

$$\dot{\hat{y}}_1(t) = H\left(\hat{y}_1, k_1, \{\bar{k}_j\}\right).$$

As it is easy to find out $H\left(\hat{y}_1, k_1, \{\bar{k}_j\}\right)$, we thus can avoid solving (18) (except using it to determine the initial value of $\hat{y}_1(t)$ when the other variables' initial values are specified). We thus can simulate a $(J+1)$-dimensional differential equations system with $\hat{y}_1(t)$, $k_1(t)$ and $\{\bar{k}(t)\}$ as the variables.

In the remainder of this study, we limit our study to the case of $\bar{\eta} = 2$. As $\xi + \lambda + \eta = 1$, we have $\eta = 1/3$. It seems that this value $\eta = 1/3$ is too high from realistic point of view. On the other hand, as our main concern of simulation is to illustrate comparative statics analysis with regard to region-specified parameters, this specification may not essentially affect our simulation conclusions. We specify this value as it enables us explicitly and uniquely solve equation (18) as follows

$$\hat{y}_1\left(k_1, \{\bar{k}_j(t)\}\right) = \Lambda_0\left(k_1, \{\bar{k}(t)\}\right) = \frac{\hat{f}}{3} + \frac{\sqrt[3]{2}\,\hat{f}^2}{3\tilde{F}} + \frac{\tilde{F}}{3\sqrt[3]{2}}, \tag{20}$$

where

$$\tilde{F}\left(k_1, \{\bar{k}_j(t)\}\right) \equiv \left[2\hat{f}^3 + 27\bar{f} + \sqrt{27\bar{f}\left(4\hat{f}^3 + 27\bar{f}\right)}\right]^{1/3}.$$

Substitute equation (20), $\bar{\phi}_1(k_1) = \alpha_1 A_1 k_1^{-\beta_1}$ and $\tilde{f}(k_1) = \alpha_1 A_1 k_1^{-\beta_1} + \delta_1$ into equation (A8#)

$$\bar{k}_1 = \Lambda\left(k_1, \{\bar{k}(t)\}\right) = \frac{\hat{f} + \sqrt[3]{2}\,\hat{f}^2/\tilde{F} + \tilde{F}/\sqrt[3]{2}}{3\alpha_1 A_1 k_1^{-\beta_1} + 3\delta_1} - \frac{\alpha_1 A_1}{\alpha_1 A_1 + \delta_1 k_1^{\beta_1}}. \tag{21}$$

According to the above analysis and equations (8), (21) and (13), we have

$$\dot{\bar{k}}_1 = \Lambda_1\left(k_1, \{\bar{k}_j\}\right) = \frac{\left(\alpha_1 A_1 k_1^{-\beta_1} + \delta_1\right)\lambda - 1}{3\left(\alpha_1 A_1 k_1^{-\beta_1} + \delta_1\right)}\left(\hat{f} + \frac{\sqrt[3]{2}\,\hat{f}^2}{\tilde{F}} + \frac{\tilde{F}}{\sqrt[3]{2}}\right) + \frac{\alpha_1 A_1}{\alpha_1 A_1 + \delta_1 k_1^{\beta_1}}, \tag{22}$$

$$\dot{\bar{k}}_j = \Lambda_j\left(k_1, \bar{k}_j\right) = \left(\alpha_1 \lambda A_1 k_1^{-\beta_j} - \bar{\lambda}\right)\bar{k}_j + \lambda A_j \beta_j \phi_j^{\alpha_j}(k_1), \quad j = 2, \cdots, J. \tag{23}$$

Taking derivatives of equation (21) with respect to t yields

$$\dot{\bar{k}}_1 = \frac{\partial \Lambda}{\partial k_1}\dot{k}_1 + \sum_{j=2}^{J}\Lambda_j \frac{\partial \Lambda}{\partial \bar{k}_j}. \tag{24}$$

Equaling two sizes of equations (22) and (24) yields

$$\dot{k}_1 = \frac{\Lambda_1\left(k_1, \{\bar{k}_j\}\right) - \sum_{j=2}^{J}\Lambda_j\left(k_1, \{\bar{k}_j\}\right)\partial\Lambda / \partial \bar{k}_j}{\partial \Lambda / \partial k_1}. \tag{25}$$

The J-dimensional differential equations (23) and (25) contain J variables, $k_1(t)$ and $\{\bar{k}(t)\}$. With proper initial conditions, the system determines the values of these variables at any point of time. It can be seen that we can determine all the other variables in the national economic system. We now simulate the model when $J = 3$.

An equilibrium point is given by setting $\dot{k}_1 = 0$ and $\dot{k}_j = 0$ in equations (22) and (23). That is

$$\frac{\left(\alpha_1 A_1 k_1^{-\beta_1} + \delta_1\right)\lambda - 1}{3\left(\alpha_1 A_1 k_1^{-\beta_1} + \delta_1\right)}\left(\hat{f}(k_1) + \frac{\sqrt[3]{2}\,\hat{f}^2(k_1)}{\tilde{F}\left(k_1, \{\bar{k}_j\}\right)} + \frac{\tilde{F}\left(k_1, \{\bar{k}_j\}\right)}{\sqrt[3]{2}}\right) + \frac{\alpha_1 A_1}{\alpha_1 A_1 + \delta_1 k_1^{\beta_1}} = 0, \tag{26}$$

$$\left(\alpha_1 \lambda A_1 k_1^{-\beta_j} - \bar{\lambda}\right)\bar{k}_j + \lambda A_j \beta_j \phi_j^{\alpha_j}(k_1) = 0, \quad j = 2, 3. \tag{27}$$

From equations (27), we solve

$$\bar{k}_j = \frac{-\lambda A_j \beta_j \phi_j^{\alpha_j}(k_1)}{\alpha_1 \lambda A_1 k_1^{-\beta_j} - \bar{\lambda}}, \quad j = 2, 3. \tag{28}$$

Substitute equations (28) into equations (13) and (19)

$$\hat{y}_j(k_1) = \frac{-\lambda A_j \beta_j \tilde{f}(k_1)\phi_j^{\alpha_j}(k_1)}{\alpha_1 \lambda_j A_1 k_1^{-\beta_j} - \bar{\lambda}} + A_j \beta_j \phi_j^{\alpha_j}(k_1), \quad j = 2, 3,$$

$$\bar{f}(k_1) = \left(\alpha_1 A_1 k_1^{-\beta_1} + \delta_1\right)\sum_{j=2}^{J}\tilde{\theta}_j\left(\phi_j(k_1) - \bar{k}_j\right)\left[\frac{-\lambda A_j \beta_j \tilde{f}(k_1)\phi_j^{\alpha_j}(k_1)}{\alpha_1 \lambda A_1 k_1^{-\beta_j} - \bar{\lambda}} + A_j \beta_j \phi_j^{\alpha_j}(k_1)\right]^{\bar{\eta}}. \tag{29}$$

Substituting equations (19) and (29) into the definition of $\tilde{F}\left(k_1, \{\bar{k}_j(t)\}\right)$, we solve $\tilde{F}$ as a function of k_1

$$\tilde{F}(k_1) = \left[2\hat{f}^3(k_1) + 27\bar{f}(k_1) + \sqrt{27\bar{f}(k_1)\left(4\hat{f}^3(k_1) + 27\bar{f}(k_1)\right)}\right]^{1/3}. \tag{30}$$

Insert equation (30) to equation (26)

$$\Omega(k_1) \equiv \frac{\left(\alpha_1 A_1 k_1^{-\beta_1} + \delta_1\right)\lambda - 1}{3\left(\alpha_1 A_1 k_1^{-\beta_1} + \delta_1\right)}\left(\hat{f}(k_1) + \frac{\sqrt[3]{2}\,\hat{f}^2(k_1)}{\tilde{F}(k_1)} + \frac{\tilde{F}(k_1)}{\sqrt[3]{2}}\right) + \frac{\alpha_1 A_1}{\alpha_1 A_1 + \delta_1 k_1^{\beta_1}} = 0. \tag{31}$$

Equation (31) determines the equilibrium value of k_1 and equations (28) determine the equilibrium values of $\bar{k}_j$, $j = 2, 3.$ From simulation with some specified values of the parameters in equation (31), we can demonstrate that $\Omega(k_1)$ decreases in k_1. Nevertheless, it is tedious to explicitly demonstrate this property because the expression is too complicated, even though the calculation is straightforward.

To simulate the model, we specify the parameter values as follows

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 1.2 \\ 1 \end{pmatrix}, \ \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3.8 \\ 4.3 \end{pmatrix}, \ \begin{pmatrix} \eta \\ \xi \\ \lambda \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/9 \\ 4/9 \end{pmatrix}, \ \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 3.5 \end{pmatrix}, \ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.3 \\ 0.3 \end{pmatrix}, \ \begin{pmatrix} \delta_{k1} \\ \delta_{k2} \\ \delta_{k3} \end{pmatrix} = \begin{pmatrix} 0.15 \\ 0.14 \\ 0.13 \end{pmatrix}, \tag{32}$$

and $N = 10$. Region 1 has the highest level of productivity and its amenity level is lower than region 3 but higher than region 2. Region 2's level of productivity is the second, next to region 1's. We term region 1 as the advanced region, region 2 the middle region, and region 3 the remote region. We now show that the dynamic system has a unique equilibrium point. It should be remarked that although the specified values are not based on empirical observations, the choice does not seem to be unrealistic. For instance, some empirical studies on the US economy demonstrate that the value of the parameter, α, in the Cobb-Douglas production is approximately equal to 0.3 (for instance, Abel and Bernanke, 1998). With regard to the technological parameters, what are important in our interregional study are their relative values. The presumed productivity differences among the regions are not very large. This is similarly true for the specified differences in the amenity parameters among regions.

We plot the function $\Omega(k_1)$ as in Figure 1. The equation $\Omega(k_1) = 0$ has a unique meaningful solution given as follows

$$k_1 = 1.118.$$

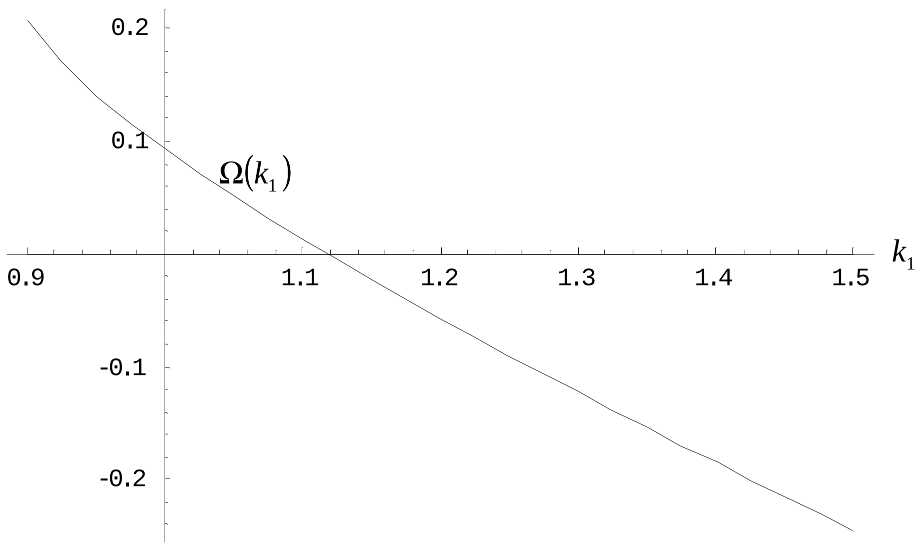


Figure 1. The Existence of Positive Solutions of $\Omega(k_1) = 0$

From $k_1 = 1.118$ and Eqs. (28), we get the equilibrium values of $\bar{k}_2$ and $\bar{k}_3$. Following Lemma 1, we get the equilibrium values of all the other variables. We list the simulation results as follows

$$F = 12.73, \ \ K = 9.06, \ \ r = 0.266,$$

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 1.551 \\ 1.116 \\ 0.852 \end{pmatrix}, \ \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 1.117 \\ 0.786 \\ 0.586 \end{pmatrix}, \ \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} = \begin{pmatrix} 4.93 \\ 2.92 \\ 2.15 \end{pmatrix}, \ \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} 7.643 \\ 3.257 \\ 1.835 \end{pmatrix}, \ \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 2.719 \\ 1.159 \\ 0.653 \end{pmatrix},$$

$$\begin{pmatrix} \bar{k}_1 \\ \bar{k}_2 \\ \bar{k}_3 \end{pmatrix} = \begin{pmatrix} 1.104 \\ 0.794 \\ 0.606 \end{pmatrix}, \ \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 1.086 \\ 0.781 \\ 0.596 \end{pmatrix}, \ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0.552 \\ 0.397 \\ 0.303 \end{pmatrix}, \ \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} 0.609 \\ 1.371 \\ 1.625 \end{pmatrix}, \ \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} = \begin{pmatrix} 1.360 \\ 0.435 \\ 0.280 \end{pmatrix}. \tag{33}$$

We see that the per-capita levels of wealth and consumption and wage rate in the advanced region higher than the corresponding variables in the middle and remote regions. But the lot size of the remote region is largest in the nation and the land rent in the region is lowest; the next is the middle region; a household in the advanced region has the smallest lot size and the land rent in the region is highest. From the interpretations of the given values of the parameters, the simulation results are expectable. We see that although workers can earn more money in the advanced region than in the remote region, they have to pay much higher rent for housing than the households in the remote region. Expectably, the typical household in the advanced region consume more goods and live in a smaller house than the typical household in the remote region. Mainly because of its technological advantage, the advanced region attracts far more workers and capital than the remote region. As the remote region has better living environment and more land than the advanced region, we can suggest that the aggregated regional economic power is mainly determined by technology. This observation theoretically demonstrates that good natural conditions and large regional endowments may play a small role in guaranteeing regional economic power in freely competitive national economy.

The output, wealth, population, and wage income distribution among regions are given by

$$\begin{pmatrix} \hat{F}_1 \\ \hat{F}_2 \\ \hat{F}_3 \end{pmatrix} = \begin{pmatrix} 60.0\% \\ 25.6\% \\ 14.4\% \end{pmatrix}, \ \begin{pmatrix} \hat{K}_1 \\ \hat{K}_2 \\ \hat{K}_3 \end{pmatrix} = \begin{pmatrix} 60.0\% \\ 25.6\% \\ 14.4\% \end{pmatrix}, \ \begin{pmatrix} \hat{N}_1 \\ \hat{N}_2 \\ \hat{N}_3 \end{pmatrix} = \begin{pmatrix} 49.3\% \\ 29.2\% \\ 21.5\% \end{pmatrix}, \ \begin{pmatrix} \hat{W}_1 \\ \hat{W}_2 \\ \hat{W}_3 \end{pmatrix} = \begin{pmatrix} 60.0\% \\ 25.6\% \\ 14.4\% \end{pmatrix}, \tag{34}$$

where a variable x_j with circumflex, $\hat{x}_j$, denotes region j's share of the corresponding variable in the national economy. The shares of the output, capital stocks, population, and wage income of the advanced region in the national economy are respectively 60.0%, 60.0%, 49.3%, and 60.0%. The remote region's corresponding shares are respectively 14.4%, 14.4%, 21.5%, and 14.4%. As the population is homogeneous and people can move freely and costlessly, the remote region still attracts some people mainly because of its amenity and residential land availability.

We calculate the regional trade balances at equilibrium as follows

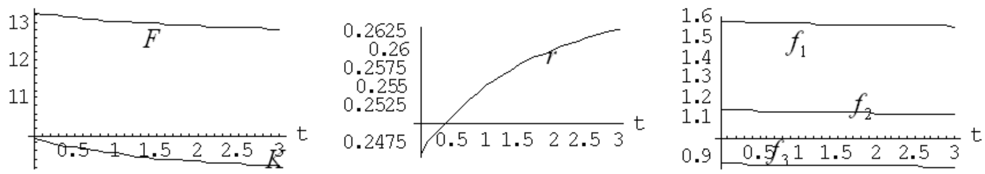
$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -0.01 \\ 0.01 \\ 0.02 \end{pmatrix}, \quad \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \begin{pmatrix} -0.07 \\ 0.03 \\ 0.04 \end{pmatrix}. \tag{35}$$

The advanced region is in trade deficit and the middle and remote regions are in trade surplus. As the population is homogenous and households have the same propensity to save, the region with the highest technology (which has the highest level of per capita capital input) tends to be in trade deficit.

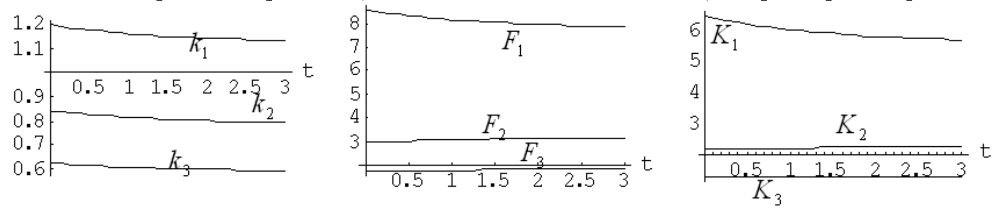
We just examined the equilibrium structure of the national economy. It is important to follow the motion of national economy when it starts from a state far away from the equilibrium. As we have shown by Lemma 1 how to follow the dynamic processes, it is straightforward to simulate the motion. We simulate the model with the parameter values specified as in (32)and the following initial conditions

$$k_1(0) = 1.2, \quad \bar{k}_2(0) = 0.75, \quad \bar{k}_3(0) = 0.6. \tag{36}$$

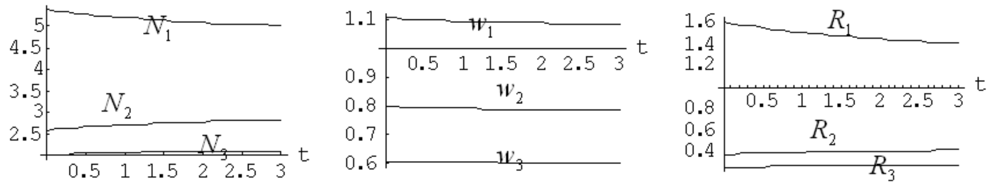
The simulation results are plotted in Figure 2. We observe that the variables approach towards their equilibrium values. It should be remarked that we don't show the convergence to the equilibrium point because the computer memory is not large enough for the task. It is significant to check stability conditions. As the calculation is too tedious, we omit the analysis.



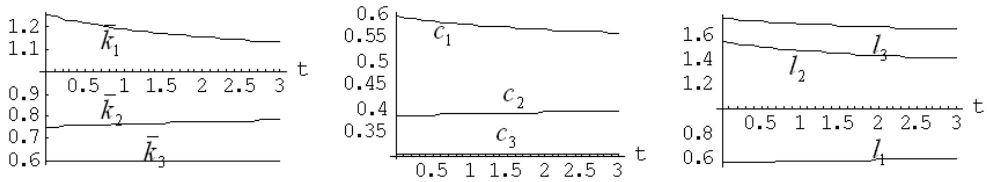
i) the national output and capital ii) the rate of interest iii) the per capita output



iv) the capital intensities v) the regional output levels vi) the regional capital levels



vii) population distribution viii) the wage rates ix) the land rents



x) the per capita wealth xi) the per capita consumption xii) the lot sizes

Figure 2. The Motion of the National Economy

4. Parameter Changes and Regional Economic Structures

It is important to ask questions such as how one region may affect the national economy as its technology or amenity is improved; or how the regional trade patterns may be affected as the propensity or the total population to save is increased. This section examines impact of changes in some parameters on the national economy and regional economic structures. As the system has a unique equilibrium, we make comparative static analysis. As we have explicitly provided the procedure to simulate the motion, it is straightforward to make comparative dynamic analysis.

First, we examine the case that all the parameters, except region 1's productivity, A_1, are the same as in (32). We increase the productivity level A_1 from 1.5 to 1.7. The simulation results are summarized in (39#), in which a variable $\bar{\Delta} x_j$ stand for the change rate of the equilibrium value of the variable x_j in percentage due to the change in the parameter value from A_{10} $(=1.5$ in this case) to A_1 $(=1.7$). That is

$$\bar{\Delta} x_j \equiv \frac{x_j(A_1) - x_j(A_{10})}{\left| x_j(A_{10}) \right|} \times 100, \tag{37}$$

where $x_j(A_1)$ stands for the equilibrium value of the variable x_j with the parameter value A_1 and $x_j(A_{10})$ stands for the value of the variable x_j with the parameter value A_{10}. We will use the symbol $\bar{\Delta}$ with the same meaning when we analyze other parameters.

We see that as the advanced region's productivity is improved, the national output is increased and the rate of interest is increased. The per-capita output level of the advanced region rises; the levels of per-capita output of the other two regions fall. The advanced region's per-capita output level rises but the other two regions' per-capita output levels fall. Some people migrate from the middle and remote regions to the advanced regions. The per-capita wealth, wage rate, and consumption level are increased in the advanced regions; the per-capita wealth and consumption levels of the other regions are almost not affected and the wage rates are reduced. The lot size of the advanced region falls and the land rent rises due to the immigration; the lot sizes of the other regions rise and the land rent fall due to emigration. The share of national product of the advanced region is further increased and the other two regions' shares are reduced. The trade balances of the advanced and middle regions improve and the remote region's trade balance deteriorates.

$$A_1 : 1.5 \Rightarrow 1.7 \Rightarrow \quad \overline{\Delta} F = 17.54, \ \overline{\Delta} K = 17.67, \ \ \overline{\Delta} r = 0.42,$$

$$\begin{pmatrix} \overline{\Delta} f_1 \\ \overline{\Delta} f_2 \\ \overline{\Delta} f_3 \end{pmatrix} = \begin{pmatrix} 19.44 \\ -0.11 \\ -0.11 \end{pmatrix}, \ \begin{pmatrix} \Delta k_1 \\ \Delta k_2 \\ \Delta k_3 \end{pmatrix} = \begin{pmatrix} 19.12 \\ -0.37 \\ -0.37 \end{pmatrix}, \ \begin{pmatrix} \overline{\Delta} N_1 \\ \overline{\Delta} N_2 \\ \overline{\Delta} N_3 \end{pmatrix} = \begin{pmatrix} 17.99 \\ -17.48 \\ -17.47 \end{pmatrix}, \ \begin{pmatrix} \overline{\Delta} F_1 \\ \overline{\Delta} F_2 \\ \overline{\Delta} F_3 \end{pmatrix} = \begin{pmatrix} 40.93 \\ -17.57 \\ -17.57 \end{pmatrix},$$

$$\begin{pmatrix} \overline{\Delta} \bar{k}_1 \\ \overline{\Delta} \bar{k}_2 \\ \overline{\Delta} \bar{k}_3 \end{pmatrix} = \begin{pmatrix} 19.58 \\ 0.00 \\ 0.00 \end{pmatrix}, \ \begin{pmatrix} \overline{\Delta} w_1 \\ \overline{\Delta} w_2 \\ \overline{\Delta} w_3 \end{pmatrix} = \begin{pmatrix} 19.44 \\ -0.11 \\ -0.11 \end{pmatrix}, \ \begin{pmatrix} \overline{\Delta} c_1 \\ \overline{\Delta} c_2 \\ \overline{\Delta} c_3 \end{pmatrix} = \begin{pmatrix} 19.58 \\ 0.00 \\ 0.01 \end{pmatrix}, \ \begin{pmatrix} \overline{\Delta} l_1 \\ \overline{\Delta} l_2 \\ \overline{\Delta} l_3 \end{pmatrix} = \begin{pmatrix} -15.25 \\ 21.18 \\ 20.18 \end{pmatrix},$$

$$\begin{pmatrix} \overline{\Delta} R_1 \\ \overline{\Delta} R_2 \\ \overline{\Delta} R_3 \end{pmatrix} = \begin{pmatrix} 41.09 \\ -17.48 \\ -17.48 \end{pmatrix}, \ \begin{pmatrix} \overline{\Delta} \hat{F}_1 \\ \overline{\Delta} \hat{F}_2 \\ \overline{\Delta} \hat{F}_3 \end{pmatrix} = \begin{pmatrix} 19.90 \\ -29.87 \\ -29.87 \end{pmatrix}, \ \begin{pmatrix} \overline{\Delta} E_1 \\ \overline{\Delta} E_2 \\ \overline{\Delta} E_3 \end{pmatrix} = \begin{pmatrix} 1.68 \\ 10.42 \\ -8.71 \end{pmatrix}. \tag{38}$$

We summarize the effects of change in the middle region's productivity in (39). The productivity level, A_2 rises from 1.2 to 1.4. As the middle region's productivity is improved, the national output is increased. The per-capita output of all the regions rise. But different from the previous case when the advanced region improves its productivity, the rate of interest falls as the middle region improves its productivity. Also different from the previous case, the middle region's per-capita output level as well as the other regions' output levels rise. Some people migrate from the advanced and remote regions to the middle region. The per-capita wealth, wage rate, and consumption level are increased in the advanced regions; the per-capita wealth and consumption levels of the other regions are almost not affected and the wage rates are increased. The lot size of the middle region falls and the land rent rises due to the immigration; the lot sizes of the other regions rise and the land rent fall due to emigration. The share of national product of the middle region is increased and the other two regions' shares are reduced. As in the previous case, the trade balances of the advanced and remote regions deteriorate and the middle region's trade balance improves.

$$A_2 : 1.2 \Rightarrow 1.4 \Rightarrow \quad \overline{\Delta} F = 6.78, \ \overline{\Delta} K = 6.71, \ \ \overline{\Delta} r = -0.24,$$

$$\begin{pmatrix} \overline{\Delta} f_1 \\ \overline{\Delta} f_2 \\ \overline{\Delta} f_3 \end{pmatrix} = \begin{pmatrix} 0.06 \\ 24.71 \\ 0.06 \end{pmatrix}, \ \begin{pmatrix} \Delta k_1 \\ \Delta k_2 \\ \Delta k_3 \end{pmatrix} = \begin{pmatrix} 0.22 \\ 24.90 \\ 0.21 \end{pmatrix}, \ \begin{pmatrix} \overline{\Delta} N_1 \\ \overline{\Delta} N_2 \\ \overline{\Delta} N_3 \end{pmatrix} = \begin{pmatrix} -13.90 \\ 33.74 \\ -13.91 \end{pmatrix}, \ \begin{pmatrix} \overline{\Delta} F_1 \\ \overline{\Delta} F_2 \\ \overline{\Delta} F_3 \end{pmatrix} = \begin{pmatrix} -13.84 \\ 66.79 \\ -13.85 \end{pmatrix},$$

$$\begin{pmatrix} \overline{\Delta} \bar{k}_1 \\ \overline{\Delta} \bar{k}_2 \\ \overline{\Delta} \bar{k}_3 \end{pmatrix} = \begin{pmatrix} 0.00 \\ 24.63 \\ 0.00 \end{pmatrix}, \ \begin{pmatrix} \overline{\Delta} w_1 \\ \overline{\Delta} w_2 \\ \overline{\Delta} w_3 \end{pmatrix} = \begin{pmatrix} 0.06 \\ 24.71 \\ 0.06 \end{pmatrix}, \ \begin{pmatrix} \overline{\Delta} c_1 \\ \overline{\Delta} c_2 \\ \overline{\Delta} c_3 \end{pmatrix} = \begin{pmatrix} 0.00 \\ 24.63 \\ 0.00 \end{pmatrix}, \ \begin{pmatrix} \overline{\Delta} l_1 \\ \overline{\Delta} l_2 \\ \overline{\Delta} l_3 \end{pmatrix} = \begin{pmatrix} 16.14 \\ -25.23 \\ 16.15 \end{pmatrix},$$

$$\begin{pmatrix} \overline{\Delta} R_1 \\ \overline{\Delta} R_2 \\ \overline{\Delta} R_3 \end{pmatrix} = \begin{pmatrix} -13.90 \\ 66.69 \\ -13.91 \end{pmatrix}, \quad \begin{pmatrix} \overline{\Delta} \hat{F}_1 \\ \overline{\Delta} \hat{F}_2 \\ \overline{\Delta} \hat{F}_3 \end{pmatrix} = \begin{pmatrix} -19.31 \\ 56.21 \\ -19.32 \end{pmatrix}, \quad \begin{pmatrix} \overline{\Delta} E_1 \\ \overline{\Delta} E_2 \\ \overline{\Delta} E_3 \end{pmatrix} = \begin{pmatrix} -0.78 \\ 34.99 \\ -19.05 \end{pmatrix}. \tag{39}$$

We summarize the effects of change in the remote region's productivity in (39). As the remote region's productivity is improved, the national output is increased and the rate of interest falls. The per-capita output levels of all the regions rise. Some people migrate from the advanced and middle regions to the remote region. The per-capita wealth, wage rate, and consumption level are increased in the remote region; the per-capita wealth and consumption levels of the other regions are almost not affected and the wage rates are increased. The lot size of the remote region falls and the land rent rises due to the immigration; the lot sizes of the other regions rise and the land rent fall due to emigration. The share of national product of the remote region is increased and the other two regions' shares are reduced. The trade balance of the remote region improves and the middle and advanced regions' trade balances deteriorate.

$$A_3 : 1 \Rightarrow 1.2 \Rightarrow \ \overline{\Delta} F = 2.25, \ \overline{\Delta} K = 2.46, \ \overline{\Delta} r = -1.11,$$

$$\begin{pmatrix} \overline{\Delta} f_1 \\ \overline{\Delta} f_2 \\ \overline{\Delta} f_3 \end{pmatrix} = \begin{pmatrix} 0.31 \\ 0.30 \\ 30.13 \end{pmatrix}, \quad \begin{pmatrix} \Delta k_1 \\ \Delta k_2 \\ \Delta k_3 \end{pmatrix} = \begin{pmatrix} 1.03 \\ 1.00 \\ 31.02 \end{pmatrix}, \quad \begin{pmatrix} \overline{\Delta} N_1 \\ \overline{\Delta} N_2 \\ \overline{\Delta} N_3 \end{pmatrix} = \begin{pmatrix} -13.39 \\ -12.41 \\ 47.45 \end{pmatrix}, \quad \begin{pmatrix} \overline{\Delta} F_1 \\ \overline{\Delta} F_2 \\ \overline{\Delta} F_3 \end{pmatrix} = \begin{pmatrix} -13.13 \\ -12.16 \\ 91.88 \end{pmatrix},$$

$$\begin{pmatrix} \overline{\Delta} \bar{k}_1 \\ \overline{\Delta} \bar{k}_2 \\ \overline{\Delta} \bar{k}_3 \end{pmatrix} = \begin{pmatrix} 0.01 \\ 1.02 \\ 31.07 \end{pmatrix}, \quad \begin{pmatrix} \overline{\Delta} w_1 \\ \overline{\Delta} w_2 \\ \overline{\Delta} w_3 \end{pmatrix} = \begin{pmatrix} 0.31 \\ 0.30 \\ 30.13 \end{pmatrix}, \quad \begin{pmatrix} \overline{\Delta} c_1 \\ \overline{\Delta} c_2 \\ \overline{\Delta} c_3 \end{pmatrix} = \begin{pmatrix} 0.01 \\ 0.57 \\ 30.48 \end{pmatrix}, \quad \begin{pmatrix} \overline{\Delta} l_1 \\ \overline{\Delta} l_2 \\ \overline{\Delta} l_3 \end{pmatrix} = \begin{pmatrix} 15.46 \\ 14.16 \\ -32.18 \end{pmatrix},$$

$$\begin{pmatrix} \overline{\Delta} R_1 \\ \overline{\Delta} R_2 \\ \overline{\Delta} R_3 \end{pmatrix} = \begin{pmatrix} -13.39 \\ -11.91 \\ 92.40 \end{pmatrix}, \quad \begin{pmatrix} \overline{\Delta} \hat{F}_1 \\ \overline{\Delta} \hat{F}_2 \\ \overline{\Delta} \hat{F}_3 \end{pmatrix} = \begin{pmatrix} -15.04 \\ -14.08 \\ 87.65 \end{pmatrix}, \quad \begin{pmatrix} \overline{\Delta} E_1 \\ \overline{\Delta} E_2 \\ \overline{\Delta} E_3 \end{pmatrix} = \begin{pmatrix} -56.70 \\ -9.86 \\ 95.30 \end{pmatrix}. \tag{40}$$

We observe that as a region improves its technology, its land rent increases and the land rents in the other regions fall. The region's trade balance is improved and the other two regions' trade balances deteriorate. It is important to note that as a region improves its technology, the other two regions' aggregated output levels fall not only in relative terms, but also in absolute values. This implies that if any region has a high rate of technological change and the other regions remain technologically stationary, then the economic activities tend to be concentrated in the technologically advancing region. Also we observe that as a region improves its technology, the other two regions' total output fall but all the individual households in the other two regions benefit in terms of consumption of goods, housing and wealth.

We summarize the effects of change in the advanced region's amenity in (41). As the advanced region's amenity is improved, the national output is increased and the rate of interest falls. The per-capita output levels and wage rates in all the regions rise. Some people migrate from the middle and remote regions to the advanced region. The per-capita wealth and consumption levels in all the regions rise. The lot size of the advanced region falls and the land rent rises; the lot sizes of the other regions rise and the land rents fall. The share of national product of the advanced region is increased and the other two regions' shares are reduced. The trade balance of the advanced region deteriorates and the middle and remote regions improve.

$$\theta_1 : 4 \Rightarrow 4.2 \Rightarrow \ \overline{\Delta} F = 1.53, \ \overline{\Delta} K = 1.83, \ \overline{\Delta} r = -0.30,$$

$$\begin{pmatrix} \overline{\Delta} f_1 \\ \overline{\Delta} f_2 \\ \overline{\Delta} f_3 \end{pmatrix} = \begin{pmatrix} 0.08 \\ 0.08 \\ 0.08 \end{pmatrix}, \ \begin{pmatrix} \Delta k_1 \\ \Delta k_2 \\ \Delta k_3 \end{pmatrix} = \begin{pmatrix} 0.28 \\ 0.27 \\ 0.27 \end{pmatrix}, \ \begin{pmatrix} \overline{\Delta} N_1 \\ \overline{\Delta} N_2 \\ \overline{\Delta} N_3 \end{pmatrix} = \begin{pmatrix} 6.84 \\ -6.63 \\ -6.65 \end{pmatrix}, \ \begin{pmatrix} \overline{\Delta} F_1 \\ \overline{\Delta} F_2 \\ \overline{\Delta} F_3 \end{pmatrix} = \begin{pmatrix} 6.93 \\ -6.57 \\ -6.57 \end{pmatrix},$$

$$\begin{pmatrix} \overline{\Delta} \bar{k}_1 \\ \overline{\Delta} \bar{k}_2 \\ \overline{\Delta} \bar{k}_3 \end{pmatrix} = \begin{pmatrix} 0.00 \\ 1.03 \\ 1.02 \end{pmatrix}, \ \begin{pmatrix} \overline{\Delta} w_1 \\ \overline{\Delta} w_2 \\ \overline{\Delta} w_3 \end{pmatrix} = \begin{pmatrix} 0.08 \\ 0.08 \\ 0.08 \end{pmatrix}, \ \begin{pmatrix} \overline{\Delta} c_1 \\ \overline{\Delta} c_2 \\ \overline{\Delta} c_3 \end{pmatrix} = \begin{pmatrix} 0.00 \\ 0.58 \\ 0.57 \end{pmatrix}, \ \begin{pmatrix} \overline{\Delta} l_1 \\ \overline{\Delta} l_2 \\ \overline{\Delta} l_3 \end{pmatrix} = \begin{pmatrix} -6.40 \\ 7.11 \\ 7.12 \end{pmatrix},$$

$$\begin{pmatrix} \overline{\Delta} R_1 \\ \overline{\Delta} R_2 \\ \overline{\Delta} R_3 \end{pmatrix} = \begin{pmatrix} 6.84 \\ -6.11 \\ -6.11 \end{pmatrix}, \ \begin{pmatrix} \overline{\Delta} \hat{F}_1 \\ \overline{\Delta} \hat{F}_2 \\ \overline{\Delta} \hat{F}_3 \end{pmatrix} = \begin{pmatrix} 5.32 \\ -8.00 \\ -8.98 \end{pmatrix}, \ \begin{pmatrix} \overline{\Delta} E_1 \\ \overline{\Delta} E_2 \\ \overline{\Delta} E_3 \end{pmatrix} = \begin{pmatrix} -30.37 \\ 57.57 \\ 14.61 \end{pmatrix}. \tag{41}$$

The effects of change in the middle region's amenity are given in (42). As the middle region's amenity is improved, the national output is reduced and the rate of interest falls. The per-capita output levels and wage rates of all the regions rise. Some people migrate from the advanced and remote regions to the middle region. The per-capita wealth and consumption levels in the advanced regions are increased and in the other two regions are only slightly affected. The lot size of the middle region falls and the land rent rises; the lot sizes of the other regions rise and the land rent fall. The share of national product of the middle region is increased and the other two regions' shares are reduced. The trade balances of the advanced and middle regions improve and the trade balance of the remote region deteriorates. The results for change in the remote region's amenity are similar to this case.

$$\theta_2 : 3.8 \Rightarrow 4 \Rightarrow \ \overline{\Delta} F = -0.56, \ \overline{\Delta} K = -0.57, \ \overline{\Delta} r = -0.05,$$

$$\begin{pmatrix} \overline{\Delta} f_1 \\ \overline{\Delta} f_2 \\ \overline{\Delta} f_3 \end{pmatrix} = \begin{pmatrix} 0.01 \\ 0.01 \\ 0.01 \end{pmatrix}, \ \begin{pmatrix} \Delta k_1 \\ \Delta k_2 \\ \Delta k_3 \end{pmatrix} = \begin{pmatrix} 0.05 \\ 0.04 \\ 0.04 \end{pmatrix}, \ \begin{pmatrix} \overline{\Delta} N_1 \\ \overline{\Delta} N_2 \\ \overline{\Delta} N_3 \end{pmatrix} = \begin{pmatrix} -4.63 \\ 11.24 \\ -4.63 \end{pmatrix}, \ \begin{pmatrix} \overline{\Delta} F_1 \\ \overline{\Delta} F_2 \\ \overline{\Delta} F_3 \end{pmatrix} = \begin{pmatrix} -4.62 \\ 11.25 \\ -4.62 \end{pmatrix},$$

$$\begin{pmatrix} \overline{\Delta} \bar{k}_1 \\ \overline{\Delta} \bar{k}_2 \\ \overline{\Delta} \bar{k}_3 \end{pmatrix} = \begin{pmatrix} 0.00 \\ 0.00 \\ 0.00 \end{pmatrix}, \ \begin{pmatrix} \overline{\Delta} w_1 \\ \overline{\Delta} w_2 \\ \overline{\Delta} w_3 \end{pmatrix} = \begin{pmatrix} 0.01 \\ 0.01 \\ 0.01 \end{pmatrix}, \ \begin{pmatrix} \overline{\Delta} c_1 \\ \overline{\Delta} c_2 \\ \overline{\Delta} c_3 \end{pmatrix} = \begin{pmatrix} 0.00 \\ 0.00 \\ 0.00 \end{pmatrix}, \ \begin{pmatrix} \overline{\Delta} l_1 \\ \overline{\Delta} l_2 \\ \overline{\Delta} l_3 \end{pmatrix} = \begin{pmatrix} 4.83 \\ -10.10 \\ 4.85 \end{pmatrix},$$

$$\begin{pmatrix} \bar{\Delta}R_1 \\ \bar{\Delta}R_2 \\ \bar{\Delta}R_3 \end{pmatrix} = \begin{pmatrix} -4.63 \\ 11.24 \\ -4.63 \end{pmatrix}, \quad \begin{pmatrix} \bar{\Delta}\hat{F}_1 \\ \bar{\Delta}\hat{F}_2 \\ \bar{\Delta}\hat{F}_3 \end{pmatrix} = \begin{pmatrix} -4.08 \\ 11.88 \\ -4.08 \end{pmatrix}, \quad \begin{pmatrix} \bar{\Delta}E_1 \\ \bar{\Delta}E_2 \\ \bar{\Delta}E_3 \end{pmatrix} = \begin{pmatrix} 1.20 \\ 6.78 \\ -5.83 \end{pmatrix}. \tag{42}$$

We conclude that if amenity improvement occurs in the technologically advanced region, then the national output rises; but if environmental improvement occurs in technologically less advanced region, the national output falls. This hints the possibility that if the central government improves a technologically less advanced region, the national output may fall. This issue will be studied in future, as this study does not include any government intervention. Environmental improvement may either improve or deteriorate the region's trade balance. It is important to note that environmental improvement has weak but symmetry impact on the regions' wage rate.

We now examine change in the preference. As our results are based on the assumption of $\eta = 1/3$, we can change on ξ and λ. As $\eta + \xi + \lambda = 1$, preference change should be subject to $\xi + \lambda = 2/3$. We now increase the propensity to save and reduce the propensity to consume as specified in (43). The effects of change in the nation's preference are provided in (43). As the propensity to save is increased, the national output is increased and the rate of interest falls. The per-capita output levels and wage rates in all the regions are increased. Some people migrate from the middle and remote regions to the advanced region. All the regions' output levels are increased. The per-capita wealth levels are all increased. The per-capita consumption levels are all reduced. The lot size of the advanced region falls; the lot sizes of the other regions rise. The land rents in all the regions rise. The share of national product of the advanced region is increased and the other two regions' shares are reduced. The trade balance of the advanced region deteriorates and the trade balances of the other two regions improve.

$$(\lambda, \xi): \left(\frac{4}{9}, \frac{2}{9}\right) \Rightarrow \left(\frac{4.3}{9}, \frac{1.7}{9}\right) \Rightarrow \ \bar{\Delta}F = 5.25, \ \bar{\Delta}K = 18.49, \ \ \bar{\Delta}r = -17.68,$$

$$\begin{pmatrix} \bar{\Delta}f_1 \\ \bar{\Delta}f_2 \\ \bar{\Delta}f_3 \end{pmatrix} = \begin{pmatrix} 5.28 \\ 5.14 \\ 5.01 \end{pmatrix}, \quad \begin{pmatrix} \Delta k_1 \\ \Delta k_2 \\ \Delta k_3 \end{pmatrix} = \begin{pmatrix} 18.70 \\ 18.19 \\ 17.71 \end{pmatrix}, \quad \begin{pmatrix} \bar{\Delta}N_1 \\ \bar{\Delta}N_2 \\ \bar{\Delta}N_3 \end{pmatrix} = \begin{pmatrix} 0.18 \\ -0.07 \\ -0.31 \end{pmatrix}, \quad \begin{pmatrix} \bar{\Delta}F_1 \\ \bar{\Delta}F_2 \\ \bar{\Delta}F_3 \end{pmatrix} = \begin{pmatrix} 5.47 \\ 5.06 \\ 4.68 \end{pmatrix},$$

$$\begin{pmatrix} \bar{\Delta}\bar{k}_1 \\ \bar{\Delta}\bar{k}_2 \\ \bar{\Delta}\bar{k}_3 \end{pmatrix} = \begin{pmatrix} 18.52 \\ 18.37 \\ 18.22 \end{pmatrix}, \quad \begin{pmatrix} \bar{\Delta}w_1 \\ \bar{\Delta}w_2 \\ \bar{\Delta}w_3 \end{pmatrix} = \begin{pmatrix} 5.28 \\ 5.14 \\ 5.01 \end{pmatrix}, \quad \begin{pmatrix} \bar{\Delta}c_1 \\ \bar{\Delta}c_2 \\ \bar{\Delta}c_3 \end{pmatrix} = \begin{pmatrix} -6.29 \\ -6.41 \\ -6.52 \end{pmatrix}, \quad \begin{pmatrix} \bar{\Delta}l_1 \\ \bar{\Delta}l_2 \\ \bar{\Delta}l_3 \end{pmatrix} = \begin{pmatrix} -0.18 \\ 0.07 \\ 0.32 \end{pmatrix},$$

$$\begin{pmatrix} \bar{\Delta}R_1 \\ \bar{\Delta}R_2 \\ \bar{\Delta}R_3 \end{pmatrix} = \begin{pmatrix} 10.45 \\ 10.03 \\ 9.63 \end{pmatrix}, \quad \begin{pmatrix} \bar{\Delta}\hat{F}_1 \\ \bar{\Delta}\hat{F}_2 \\ \bar{\Delta}\hat{F}_3 \end{pmatrix} = \begin{pmatrix} 0.21 \\ -0.18 \\ -0.54 \end{pmatrix}, \quad \begin{pmatrix} \bar{\Delta}E_1 \\ \bar{\Delta}E_2 \\ \bar{\Delta}E_3 \end{pmatrix} = \begin{pmatrix} -32.98 \\ 33.96 \\ 33.42 \end{pmatrix}. \tag{43}$$

We now allow the advanced region's residential land to be enlarged. The effects of change are given in (44). As the advanced region's residential area is enlarged, the national output is increased and the rate of interest rises. The per-capita output levels and wage rates in all the regions decline. Some people migrate from the middle and remote regions to the advanced region. The per-capita wealth and consumption levels are all only slightly affected. The lot sizes in all the regions rise; the land rents in all the regions fall. The share of national product of the advanced region is increased and the other two regions' shares are reduced. The trade balances of the advanced and middle regions improve and the remote region's trade balance deteriorates.

$$L_1 : 3 \Rightarrow 3.3 \Rightarrow \ \bar{\Delta} F = 0.67, \ \bar{\Delta} K = 0.69, \ \ \bar{\Delta} r = 0.05,$$

$$\begin{pmatrix} \bar{\Delta} f_1 \\ \bar{\Delta} f_2 \\ \bar{\Delta} f_3 \end{pmatrix} = \begin{pmatrix} -0.01 \\ -0.01 \\ -0.01 \end{pmatrix}, \ \begin{pmatrix} \Delta k_1 \\ \Delta k_2 \\ \Delta k_3 \end{pmatrix} = \begin{pmatrix} -0.05 \\ -0.05 \\ -0.05 \end{pmatrix}, \ \begin{pmatrix} \bar{\Delta} N_1 \\ \bar{\Delta} N_2 \\ \bar{\Delta} N_3 \end{pmatrix} = \begin{pmatrix} 3.27 \\ -3.18 \\ -3.18 \end{pmatrix}, \ \begin{pmatrix} \bar{\Delta} F_1 \\ \bar{\Delta} F_2 \\ \bar{\Delta} F_3 \end{pmatrix} = \begin{pmatrix} 3.26 \\ -3.19 \\ -3.19 \end{pmatrix},$$

$$\begin{pmatrix} \bar{\Delta} \bar{k}_1 \\ \bar{\Delta} \bar{k}_2 \\ \bar{\Delta} \bar{k}_3 \end{pmatrix} = \begin{pmatrix} 0.00 \\ 0.00 \\ 0.00 \end{pmatrix}, \ \begin{pmatrix} \bar{\Delta} w_1 \\ \bar{\Delta} w_2 \\ \bar{\Delta} w_3 \end{pmatrix} = \begin{pmatrix} -0.01 \\ -0.01 \\ -0.01 \end{pmatrix}, \ \begin{pmatrix} \bar{\Delta} c_1 \\ \bar{\Delta} c_2 \\ \bar{\Delta} c_3 \end{pmatrix} = \begin{pmatrix} 0.00 \\ 0.00 \\ 0.00 \end{pmatrix}, \ \begin{pmatrix} \bar{\Delta} l_1 \\ \bar{\Delta} l_2 \\ \bar{\Delta} l_3 \end{pmatrix} = \begin{pmatrix} 3.29 \\ 3.29 \\ 3.28 \end{pmatrix},$$

$$\begin{pmatrix} \bar{\Delta} R_1 \\ \bar{\Delta} R_2 \\ \bar{\Delta} R_3 \end{pmatrix} = \begin{pmatrix} -3.18 \\ -3.18 \\ -3.18 \end{pmatrix}, \ \begin{pmatrix} \bar{\Delta} \hat{F}_1 \\ \bar{\Delta} \hat{F}_2 \\ \bar{\Delta} \hat{F}_3 \end{pmatrix} = \begin{pmatrix} 2.56 \\ -3.85 \\ -3.85 \end{pmatrix}, \ \begin{pmatrix} \bar{\Delta} E_1 \\ \bar{\Delta} E_2 \\ \bar{\Delta} E_3 \end{pmatrix} = \begin{pmatrix} 0.78 \\ 1.09 \\ -1.89 \end{pmatrix}. \tag{44}$$

We now allow the middle region's residential area to be enlarged. The effects of change are given in (45). Different from the situation when the advanced region's residential area is enlarged, the national output is reduced and the rate of interest falls. The per-capita output levels and wage rates in all the regions are slightly affected. Some people migrate from the advanced and remote regions to the middle region. The per-capita wealth and consumption levels are all only slightly affected. The wage rates of all the regions rise. The lot sizes in all the regions rise; the land rents in all the regions fall. The share of national product of the middle region is increased and the other two regions' shares are reduced. The trade balances of the advanced and middle regions improve and the remote region's trade balance deteriorates. We omit representing the effects caused by the remote region's residential area.

$$L_2 : 4 \Rightarrow 4.2 \Rightarrow \ \bar{\Delta} F = -0.17, \ \bar{\Delta} K = -0.18, \ \ \bar{\Delta} r = -0.02,$$

$$\begin{pmatrix} \bar{\Delta} f_1 \\ \bar{\Delta} f_2 \\ \bar{\Delta} f_3 \end{pmatrix} = \begin{pmatrix} 0.00 \\ 0.00 \\ 0.00 \end{pmatrix}, \ \begin{pmatrix} \Delta k_1 \\ \Delta k_2 \\ \Delta k_3 \end{pmatrix} = \begin{pmatrix} 0.01 \\ 0.01 \\ 0.01 \end{pmatrix}, \ \begin{pmatrix} \bar{\Delta} N_1 \\ \bar{\Delta} N_2 \\ \bar{\Delta} N_3 \end{pmatrix} = \begin{pmatrix} -1.44 \\ 3.49 \\ -1.44 \end{pmatrix}, \ \begin{pmatrix} \bar{\Delta} F_1 \\ \bar{\Delta} F_2 \\ \bar{\Delta} F_3 \end{pmatrix} = \begin{pmatrix} -1.43 \\ 3.49 \\ -1.43 \end{pmatrix},$$

$$\begin{pmatrix} \Delta \bar{\bar{k}}_1 \\ \Delta \bar{\bar{k}}_2 \\ \Delta \bar{\bar{k}}_3 \end{pmatrix} = \begin{pmatrix} 0.00 \\ 0.00 \\ 0.00 \end{pmatrix}, \ \begin{pmatrix} \bar{\Delta} w_1 \\ \bar{\Delta} w_2 \\ \bar{\Delta} w_3 \end{pmatrix} = \begin{pmatrix} 0.00 \\ 0.00 \\ 0.00 \end{pmatrix}, \ \begin{pmatrix} \bar{\Delta} c_1 \\ \bar{\Delta} c_2 \\ \bar{\Delta} c_3 \end{pmatrix} = \begin{pmatrix} 0.00 \\ 0.00 \\ 0.00 \end{pmatrix}, \ \begin{pmatrix} \bar{\Delta} l_1 \\ \bar{\Delta} l_2 \\ \bar{\Delta} l_3 \end{pmatrix} = \begin{pmatrix} 1.46 \\ 1.46 \\ 1.46 \end{pmatrix},$$

$$\begin{pmatrix} \bar{\Delta} R_1 \\ \bar{\Delta} R_2 \\ \bar{\Delta} R_3 \end{pmatrix} = \begin{pmatrix} -1.44 \\ -1.44 \\ -1.44 \end{pmatrix}, \ \begin{pmatrix} \bar{\Delta} \hat{F}_1 \\ \bar{\Delta} \hat{F}_2 \\ \bar{\Delta} \hat{F}_3 \end{pmatrix} = \begin{pmatrix} -1.26 \\ 3.67 \\ -1.26 \end{pmatrix}, \ \begin{pmatrix} \bar{\Delta} E_1 \\ \bar{\Delta} E_2 \\ \bar{\Delta} E_3 \end{pmatrix} = \begin{pmatrix} 0.34 \\ 2.21 \\ -1.82 \end{pmatrix}. \tag{45}$$

Finally, we allow the population to increase from 10 to 11. The national output and each region's output levels are increased. The per-capita output levels and wage rates in all the regions are increased.

$$N: 10 \Rightarrow 11 \Rightarrow \ \bar{\Delta} F = 10.09, \ \bar{\Delta} K = 10.53, \ \bar{\Delta} r = -0.64,$$

$$\begin{pmatrix} \bar{\Delta} f_1 \\ \bar{\Delta} f_2 \\ \bar{\Delta} f_3 \end{pmatrix} = \begin{pmatrix} 0.18 \\ 0.17 \\ 0.17 \end{pmatrix}, \ \begin{pmatrix} \Delta k_1 \\ \Delta k_2 \\ \Delta k_3 \end{pmatrix} = \begin{pmatrix} 0.59 \\ 0.58 \\ 0.56 \end{pmatrix}, \ \begin{pmatrix} \bar{\Delta} N_1 \\ \bar{\Delta} N_2 \\ \bar{\Delta} N_3 \end{pmatrix} = \begin{pmatrix} 9.54 \\ 10.45 \\ 10.45 \end{pmatrix}, \ \begin{pmatrix} \bar{\Delta} F_1 \\ \bar{\Delta} F_2 \\ \bar{\Delta} F_3 \end{pmatrix} = \begin{pmatrix} 9.73 \\ 10.64 \\ 10.63 \end{pmatrix},$$

$$\begin{pmatrix} \Delta \bar{\bar{k}}_1 \\ \Delta \bar{\bar{k}}_2 \\ \Delta \bar{\bar{k}}_3 \end{pmatrix} = \begin{pmatrix} 0.28 \\ 1.02 \\ 1.02 \end{pmatrix}, \ \begin{pmatrix} \bar{\Delta} w_1 \\ \bar{\Delta} w_2 \\ \bar{\Delta} w_3 \end{pmatrix} = \begin{pmatrix} 0.18 \\ 0.18 \\ 0.17 \end{pmatrix}, \ \begin{pmatrix} \bar{\Delta} c_1 \\ \bar{\Delta} c_2 \\ \bar{\Delta} c_3 \end{pmatrix} = \begin{pmatrix} 0.16 \\ 0.57 \\ 0.57 \end{pmatrix}, \ \begin{pmatrix} \bar{\Delta} l_1 \\ \bar{\Delta} l_2 \\ \bar{\Delta} l_3 \end{pmatrix} = \begin{pmatrix} -8.71 \\ -9.46 \\ -9.46 \end{pmatrix},$$

$$\begin{pmatrix} \bar{\Delta} R_1 \\ \bar{\Delta} R_2 \\ \bar{\Delta} R_3 \end{pmatrix} = \begin{pmatrix} 9.71 \\ 11.09 \\ 11.08 \end{pmatrix}, \ \begin{pmatrix} \bar{\Delta} \hat{F}_1 \\ \bar{\Delta} \hat{F}_2 \\ \bar{\Delta} \hat{F}_3 \end{pmatrix} = \begin{pmatrix} -0.33 \\ 0.50 \\ 0.49 \end{pmatrix}, \ \begin{pmatrix} \bar{\Delta} E_1 \\ \bar{\Delta} E_2 \\ \bar{\Delta} E_3 \end{pmatrix} = \begin{pmatrix} -37.10 \\ 56.13 \\ 26.08 \end{pmatrix}.$$

It should be remarked that our model demonstrates divergence of wages rates among regions in the long term. From our simulation result, we see that although wage rates are affected by changes in regional differences technology, land and amenity, the wage disparity is mostly strongly affected by change in regional technology. This also hints that as far as neoclassical economy is concerned, if technological differences among regions are not large, wage rates may tend to converge as the other factors weakly affect the differences. It is interesting to note that from the empirical study on wage determination with a large panel of French workers, Combes *et al*. (2003) have found that individual skills account for a large fraction of existing spatial wage disparities with strong evidence of spatial sorting by skills and endowments only appear to play a small role. As far as relations between wage disparity and endowments are concerned, our model gives similar results. It remains to prove whether this character of economic development can be observed under more general conditions.

5. Spatial Equilibrium of a Two-Group and Two-Region Model

This section demonstrates that the dynamics of the two-region two-group economy can be expressed as a four dimensional differential equations system. The following lemma is proved in the appendix.

Lemma 1

The dynamics of the national economy is given by the following $4-$dimensional differential equations with $k_1(t)$ and $\bar{k}_{jq}(t), (j, q) \in \bar{Q}$, where $\bar{Q} \equiv \{(1, 2), (2, 1), (2, 2)\}$, as the variables

$$\dot{k}_1 = \left[\Lambda_{11}\left(k_1(t), \{\bar{k}_{jq}(t)\}\right) - \sum_{(j,q) \in \bar{Q}} \Lambda_{jq}(t) \frac{\partial \Lambda(t)}{\partial \bar{k}_{jq}(t)} \right] \left(\frac{\partial \Lambda(t)}{\partial k_1(t)} \right)^{-1},$$

$$\dot{\bar{k}}_{jq}(t) = \Lambda_{jq}\left(k_1(t), \{\bar{k}_{jq}(t)\}\right), \ (j, q) \in \bar{Q}, \tag{46}$$

in which $\Lambda_{jq}(t)$ and $\Lambda(t)$ are functions of $k_1(t)$ and $\{\bar{k}_{jq}(t)\}$, $(j, q) \in \bar{Q}$, defined in the appendix. For any given positive values of $k_1(t)$ and $\{\bar{k}_{jq}(t)\}$ at any point of time, the other variables are uniquely determined by the following procedure: $\bar{k}_{11}(t)$ by (A15) → $\hat{y}_{jq}(t)$ by (4), $(j, q) \in Q^*$ → $k_2(t)$ by (A1) → $f_j(t) = f_j(k_j)$, $j = 1, 2$ → $r(t)$ and $w_{jq}(t)$ by (2) → $N_{jq}(t)$ by (A9), $(j, q) \in Q^*$ → $N_j(t)$, $j = 1, 2$ by (1) → $R_j(t)$, $j = 1, 2$ by (A4) → $l_{jq}(t)$, $c_{jq}(t)$ and $s_{jq}(t)$ by (8), $(j, q) \in Q^*$ → $F_j(t) = N_j(t) f_j(t)$ → $U_{jq}(t)$ by (6), $(j, q) \in Q^*$.

Lemma 1 shows that for given initial conditions, $k_1(0)$ and $\{\bar{k}_{jq}(0)\}$, all the variables of the system can be determined. This makes the system computational. The general dynamic system is complicated. It can be seen that a thorough examination of the general dynamic system is too difficult.[2] For illustration, we simulate the model. We specify the production functions as follows

$$F_j = A_j K_j^{\alpha_j} N_j^{\beta_j}, \ \alpha_j + \beta_j = 1, \ \alpha_j, \beta_j > 0, \quad j = 1, 2,$$

where A_j is region j's productivity. From $f_j = A_j k_j^{\alpha_j}$ and equation (A1), we have

$$k_2 = \phi_2(k_1) \equiv \left(\frac{\alpha_1 A_1 k_1^{-\beta_1} - \delta_2}{\alpha_2 A_2} \right)^{-1/\beta_2}. \tag{47}$$

[2] Dynamic analyses of the economic system with identical regions and heterogeneous households are referred to Zhang (2005).

By equations (A2) and (4)

$$r = \phi_r(k_1) \equiv \alpha_1 A_1 k_1^{-\beta_1} - \delta_{k1}, \quad w_{jq} = \phi_{jq}(k_1) \equiv \beta_j z_q f_j(k_j),$$
$$\hat{y}_{jq} = (1 + \phi_r(k_1))\bar{k}_{jq} + \phi_{jq}(k_1), \tag{48}$$

We see that k_2, r and w_{jq}, $(j, q) \in Q^*$, can be expressed as unique functions of k_1.

Because a thorough dynamic analysis is too complicated, in the remainder of this study, we are only concerned with equilibrium. By equations (7) and (8), at equilibrium we have

$$\hat{y}_{jq} = \frac{\bar{k}_{jq}}{\lambda_q}, \quad (j, q) \in Q^*. \tag{49}$$

Substitute equations (49) into equations (48)

$$\bar{k}_{jq} = \frac{\phi_{jq}(k_1)}{\bar{\lambda}_q - \alpha_1 A_1 k_1^{-\beta_1}}, \quad (j, q) \in Q^*, \tag{50}$$

where $\bar{\lambda}_q \equiv 1/\lambda_q - 1 + \delta_{k1} > 0$, $q = 1, 2$. Substituting equations (49) into equations (A3), we have

$$\left(\frac{\theta_{21}}{\theta_{11}}\right)^{1/\eta_{01}} \left(\frac{\bar{k}_{21}}{\bar{k}_{11}}\right)^{\rho_1/\eta_{01}} = \left(\frac{\theta_{22}}{\theta_{12}}\right)^{1/\eta_{02}} \left(\frac{\bar{k}_{22}}{\bar{k}_{12}}\right)^{\rho_2/\eta_{02}}.$$

Insert equations (50) and (48) into the above equation

$$\left(\frac{\theta_{21}}{\theta_{11}}\right)^{1/\eta_{01}} \left(\frac{\beta_2 A_2 k_2^{\alpha_2}}{\beta_1 A_1 k_1^{\alpha_1}}\right)^{\rho_1/\eta_{01}} = \left(\frac{\theta_{22}}{\theta_{12}}\right)^{1/\eta_{02}} \left(\frac{\beta_2 A_2 k_2^{\alpha_2}}{\beta_1 A_1 k_1^{\alpha_1}}\right)^{\rho_2/\eta_{02}}. \tag{51}$$

It can be seen that if $\delta_{k1} = \delta_{k2}$ and $\alpha_1 = \alpha_2$, then the above equation does not contain k_1 and k_2.[3] For convenience of discussion, we assume $\delta_{k1} = \delta_{k2}$ (that is, $\delta_2 = 0$) and $\alpha_1 \neq \alpha_2$. From equations (47), we have

$$k_2 = \left(\frac{\alpha_1 A_1}{\alpha_2 A_2}\right)^{-1/\beta_2} k_1^{\beta_1/\beta_2}. \tag{52}$$

Insert this equation into equation (51)

$$k_1 = \left[\left(\frac{\theta_{11}}{\theta_{21}}\right)^{1/\eta_{01}} \left(\frac{\theta_{22}}{\theta_{12}}\right)^{1/\eta_{02}} \left(\frac{\alpha_1}{\alpha_2}\right)^{\alpha_2 x_0/\beta_2} \left(\frac{\beta_1}{\beta_2}\right)^{x_0} \left(\frac{A_1}{A_2}\right)^{x_0/\beta_2}\right]^{x}, \tag{53}$$

in which

[3] This means that if $\delta_{k1} = \delta_{k2}$ and $\alpha_1 = \alpha_2$, then the above conditions of utility equalization is invalid. Equilibrium may be characterized by that one group's population is entirely concentrated in one region. We neglect this case.

$$x_0 \equiv \frac{\rho_1}{\eta_{01}} - \frac{\rho_2}{\eta_{02}}, \ \ x \equiv \frac{1}{(\alpha_2 \beta_1 / \beta_2 - \alpha_1) x_0}.$$

We solve the equilibrium value of k_1 by equation (53). We solve k_2 by equation (52), $\bar{k}_{jq}$ by equations (50), $\hat{y}_{jq}$ by equations (49), and r and w_{jq} by equations (48). As in equations (A8), we solve the labor distribution by the following four linear equations

$$\begin{aligned} &\bar{y}_{11} N_{11} + \bar{y}_{12} N_{12} + \bar{y}_{21} N_{21} + \bar{y}_{22} N_{22} = 0, \\ &f_{11} N_{11} + f_{12} N_{12} + f_{21} N_{21} + f_{22} N_{22} = 0, \\ &N_{11} + N_{21} = \bar{N}_1, \\ &N_{12} + N_{22} = \bar{N}_2, \end{aligned} \tag{54}$$

in which

$$\bar{y}_{1q} \equiv \frac{\eta_q \bar{k}_{1q} L_2}{\lambda_q L_1} \left(\frac{\theta_{2q}}{\theta_{1q}} \right)^{1/\eta_{0q}} \left(\frac{\bar{k}_{2q}}{\bar{k}_{1q}} \right)^{\rho_q / \eta_{0q}}, \ \ \bar{y}_{2q} \equiv - \frac{\eta_q}{\lambda_q} \bar{k}_{2q}, \ \ q = 1, 2,$$

$$f_{jq}(k_1) \equiv \left(1 + \frac{\xi_q}{\lambda_q} \right) \bar{k}_{jq} - \left(f_j + \bar{\delta}_j k_j \right) z_q, \ \ (j, q) \in Q^*.$$

As $\bar{N}_j$, f_{jq} and $\bar{y}_{jq}$ are parameters now, the four equations (54) have 4 variables and are linear in the variables. Applying Cramer's rule, we solve the variables, $N_{jq}, (j, q) \in Q^*$. Hence we solve the labor distribution. We solve the equilibrium values of the other variables as in Lemma 1: N_j by (1) $\rightarrow$ R_j by (A4) $\rightarrow$ l_{jq}, c_{jq} and s_{jq} by (8) $\rightarrow$ $F_j = N_j f_j$ $\rightarrow$ U_{jq} by (6).

As the procedure for simulation is explicitly provided, we can simulate the model. As the expressions are too tedious, we will not examine dynamic properties of the model here.

6. Conclusions

This paper proposed a multi-region growth model with heterogeneous households and capital accumulation. Different from the growth models with the Ramsey approach in the literature of economic growth, we used a utility function, which determines saving and consumption with utility optimization without leading to a higher dimensional dynamic system like by the Ramsey approach. It should be remarked that this paper is a generalization of the two-region growth model proposed by Zhang (1998). Some multi-region models by the approach are given in Zhang (2007, 2008). The previous studies are all concerned with a homogeneous population. It is well known that one-sector growth model has been generalized and extended in many directions. It is not difficult to generalize our model along these lines (see Zhang, 2005).

Appendix Proving Lemma 1

We now show that the dynamics of the national economy can be expressed as 4 differential equations. First, from equations (2) we obtain

$$f_2^{'}(k_2) = f_1^{'}(k_1) - \delta_2, \tag{A1}$$

where $\delta_2 \equiv \delta_{k1} - \delta_{k2}$. If $f_1^{'}(k_1) - \delta_2 > 0$ and given $k_1(t) > 0$, then the equation determines a unique relation between k_2 and k_1, denoted by $k_2 = \phi_2(k_1)$. From equation (A1), we have

$$f_2^{''}(k_2)\frac{dk_2}{dk_1} = f_1^{''}(k_1).$$

As $f_2^{''}(k_2) \leq 0$, we see that $dk_2 / dk_1 \geq 0$. That is, $\phi^{'}(k_1) \geq 0$. Hence, for any given $k_1(t) > 0$, we determine $k_2(t)$ as a unique function of $k_1(t)$. From equations (2), we determine the wage rates as functions of $k_1(t)$ as follows

$$w_{jq}(t) = \phi_{jq}(k_1) \equiv z_q\left\{f_j(\phi_j(k_1)) - \phi_j(k_1) f_j^{'}(\phi_j(k_1))\right\}, \tag{A2}$$

in which $\phi_1(k_1) = k_1$. We see that the capital intensity, $k_2(t)$, the wage rates, $w_{jq}(t)$, the per-worker output levels, $f_j(t)$, and the rate of interest, $r(t)$, can be expressed as functions of $k_1(t)$.

Substituting equations (7) into (6) and then applying equations (9), we have

$$\frac{R_2}{R_1} = \left(\frac{\theta_{2q}}{\theta_{1q}}\right)^{1/\eta_{0q}}\left(\frac{\hat{y}_{2q}}{\hat{y}_{1q}}\right)^{\rho_q/\eta_{0q}}, \quad q = 1, 2. \tag{A3}$$

From equations (A3), we have

$$\left(\frac{\theta_{21}}{\theta_{11}}\right)^{1/\eta_{01}}\left(\frac{\hat{y}_{21}}{\hat{y}_{11}}\right)^{\rho_1/\eta_{01}} = \left(\frac{\theta_{22}}{\theta_{12}}\right)^{1/\eta_{02}}\left(\frac{\hat{y}_{22}}{\hat{y}_{12}}\right)^{\rho_2/\eta_{02}}.$$

Substituting $l_{jq} = \eta_q \hat{y}_{jq} / R_j$ into the land constraint in (A1), we have

$$R_j = \frac{\eta_1 \hat{y}_{j1} N_{j1} + \eta_2 \hat{y}_{j2} N_{j2}}{L_j}. \tag{A4}$$

Insert the above equations into equations (A3)

$$\frac{\eta_1 \hat{y}_{21} N_{21} + \eta_2 \hat{y}_{22} N_{22}}{\eta_1 \hat{y}_{11} N_{11} + \eta_2 \hat{y}_{12} N_{12}} = \frac{L_2}{L_1}\left(\frac{\theta_{2q}}{\theta_{1q}}\right)^{1/\eta_{0q}}\left(\frac{\hat{y}_{2q}}{\hat{y}_{1q}}\right)^{\rho_q/\eta_{0q}}, \quad q = 1, 2.$$

We can rewrite the above equation as

$$\bar{y}_{11} N_{11} + \bar{y}_{12} N_{12} + \bar{y}_{21} N_{21} + \bar{y}_{22} N_{22} = 0, \tag{A5}$$

in which

$$\overline{y}_{1q}\left(\hat{y}_{1q}, \hat{y}_{2q}\right) \equiv \overline{\eta}_q \left(\frac{\hat{y}_{2q}}{\hat{y}_{1q}}\right)^{\rho_q / \eta_{0q}}, \ \overline{y}_{2q}\left(\hat{y}_{2q}\right) \equiv -\eta_q \hat{y}_{2q}, \ \overline{\eta}_q \equiv \frac{\eta_q \hat{y}_{1q} L_2}{L_1}\left(\frac{\theta_{2q}}{\theta_{1q}}\right)^{1/\eta_{0q}}, \ q = 1, 2.$$

Substituting equations (7) into equation (11), we obtain

$$\sum_{j=1}^{2}\sum_{q=1}^{2}\left(\xi_q + \lambda_q\right)\hat{y}_{jq} N_{jq} = F_1 + F_2 + \overline{\delta}_1 K_1 + \overline{\delta}_2 K_2, \tag{A6}$$

where $\overline{\delta}_j \equiv 1 - \delta_{kj}$. Substituting

$$F_j = \left(z_1 N_{j1} + z_2 N_{j2}\right) f_j\left(k_j\right), \ K_j = \left(z_1 N_{j1} + z_2 N_{j2}\right) k_j,$$

into equation (A6) yields

$$\sum_{j=1}^{2}\sum_{q=1}^{2}\left(\xi_q + \lambda_q\right)\hat{y}_{jq} N_{jq} = \left(f_1 + \overline{\delta}_1 k_1\right)\left(z_1 N_{j1} + z_2 N_{j2}\right) + \left(f_2 + \overline{\delta}_2 k_2\right)\left(z_1 N_{21} + z_2 N_{22}\right).$$

We can rewrite the above equation as follows

$$f_{11} N_{11} + f_{12} N_{12} + f_{21} N_{21} + f_{22} N_{22} = 0, \tag{A7}$$

in which

$$f_{jq}\left(k_1, \hat{y}_{jq}\right) \equiv \left(\xi_q + \lambda_q\right)\hat{y}_{jq} - \left(f_j + \overline{\delta}_j k_j\right) z_q, \quad j, q = 1, 2.$$

For any given k_1 and $\left\langle \hat{y}_{jq}(t)\right\rangle$, where

$$\left\langle \hat{y}_{jq}(t)\right\rangle = \left[\hat{y}_{11}(t)\ \hat{y}_{12}(t)\ \hat{y}_{21}(t)\ \hat{y}_{22}(t)\right],$$

the four equations, (A5), (A6) and the two labor constraints in (12), contain four variables, $\left\langle N_{jq}(t)\right\rangle$. The equations are linear in $N_{jq}(t)$. We can rewrite these four equations in the following matrix form

$$\begin{bmatrix} \overline{y}_{11}(t) & \overline{y}_{12}(t) & \overline{y}_{21}(t) & \overline{y}_{22}(t) \\ f_{11}(t) & f_{12}(t) & f_{21}(t) & f_{22}(t) \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} N_{11}(t) \\ N_{12}(t) \\ N_{21}(t) \\ N_{22}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \overline{N}_1 \\ \overline{N}_2 \end{bmatrix}. \tag{A8}$$

Applying Cramer's rule, we solve N_{jq} as follows

$$N_{jq}(t) = \frac{\Lambda_{jq}\left(k_1(t), \left\langle \hat{y}_{jq}(t)\right\rangle\right)}{\Lambda\left(k_1(t), \left\langle \hat{y}_{jq}(t)\right\rangle\right)}, \tag{A9}$$

in which

$$\Lambda\left(k_1, \left\langle \hat{y}_{jq}\right\rangle\right) \equiv f_{12}\overline{y}_{11} - f_{22}\overline{y}_{11} - f_{11}\overline{y}_{12} + f_{21}\overline{y}_{12} - f_{12}\overline{y}_{21} + f_{22}\overline{y}_{21} + f_{11}\overline{y}_{22} - f_{21}\overline{y}_{22},$$

$$\Lambda_{11}\left(k_1, \left\langle \hat{y}_{jq}\right\rangle\right) \equiv \overline{N}_1\left(f_{21}\overline{y}_{12} - f_{12}\overline{y}_{21} + f_{22}\overline{y}_{21} - f_{21}\overline{y}_{22}\right) + \overline{N}_2\left(f_{22}\overline{y}_{12} - f_{12}\overline{y}_{22}\right),$$

$$\Lambda_{12}\left(k_1, \left\langle \hat{y}_{jq}\right\rangle\right) \equiv -\overline{N}_1\left(f_{21}\overline{y}_{11} - f_{11}\overline{y}_{21}\right) - \overline{N}_2\left(f_{22}\overline{y}_{11} - f_{22}\overline{y}_{21} - f_{11}\overline{y}_{22} + f_{21}\overline{y}_{22}\right),$$

$$\Lambda_{21}\left(k_1, \left\langle \hat{y}_{jq}\right\rangle\right) \equiv \overline{N}_1\left(f_{12}\overline{y}_{11} - f_{22}\overline{y}_{11} - f_{11}\overline{y}_{12} + f_{11}\overline{y}_{22}\right) - \overline{N}_2\left(f_{22}\overline{y}_{12} - f_{12}\overline{y}_{22}\right),$$

$$\Lambda_{22}\left(k_1, \left\langle \hat{y}_{jq} \right\rangle\right) \equiv \bar{N}_1\left(f_{21}\bar{y}_{11} - f_{11}\bar{y}_{21}\right) + \bar{N}_2\left(f_{12}\bar{y}_{11} - f_{11}\bar{y}_{12} + f_{21}\bar{y}_{12} - f_{12}\bar{y}_{21}\right).$$

From equation (10), we have

$$\left(\bar{k}_{11} - z_1 k_1\right)N_{11} + \left(\bar{k}_{12} - z_2 k_1\right)N_{12} + \left(\bar{k}_{21} - z_1 k_2\right)N_{21} + \left(\bar{k}_{22} - z_2 k_2\right)N_{22} = 0.$$

Insert equations (A9) into the above equation

$$\left(\bar{k}_{11} - z_1 k_1\right)\Lambda_{11} + \bar{K}_{12}\Lambda_{12} + \bar{K}_{21}\Lambda_{21} + \bar{K}_{22}\Lambda_{22} = 0, \tag{A10}$$

in which $\bar{K}_{jq} \equiv \bar{k}_{jq} - z_q k_j$.

From the definitions of f_{jq} and $\bar{y}_{jq}$, we see that only f_{11} and $\bar{y}_{11}$ contain $\hat{y}_{11}$ and the rest of f_{jq} and $\bar{y}_{jq}$ are independent of f_{11} and $\bar{y}_{11}$. Substituting Λ_{jq} into the above equation, we get

$$\Lambda_{11}\bar{k}_{11} - \Lambda_1\bar{y}_{11} - \Lambda_2 f_{11} + \Lambda_3 = 0, \tag{A11}$$

in which

$$\Lambda_1\left(k_1(t), \left\{\hat{y}_{jq}(t)\right\}\right) \equiv f_{21}\bar{K}_{12}\bar{N}_1 + f_{22}\bar{K}_{12}\bar{N}_2 - f_{12}\bar{K}_{21}\bar{N}_1 + f_{22}\bar{K}_{21}\bar{N}_1 - f_{21}\bar{K}_{22}\bar{N}_1 - f_{12}\bar{K}_{22}\bar{N}_2,$$

$$\Lambda_2\left(k_1(t), \left\{\hat{y}_{jq}(t)\right\}\right) \equiv -\bar{y}_{21}\bar{K}_{12}\bar{N}_1 - \bar{y}_{22}\bar{K}_{12}\bar{N} + \bar{y}_{12}\bar{K}_{21}\bar{N}_1 - \bar{y}_{22}\bar{K}_{21}\bar{N}_1 + \bar{y}_{21}\bar{K}_{22}\bar{N}_1 + \bar{y}_{12}\bar{K}_{22}\bar{N}_2,$$

$$\Lambda_2\left(k_1(t), \left\{\hat{y}_{jq}(t)\right\}\right) \equiv$$

$$-z_1 k_1 \Lambda_{11} + \bar{K}_{12}\bar{N}\left(f_{22}\bar{y}_{21} - f_{21}\bar{y}_{22}\right) - \bar{K}_{21}\bar{N}_2\left(f_{22}\bar{y}_{12} - f_{12}\bar{y}_{22}\right) + \bar{K}_{22}\bar{N}_2\left(f_{21}\bar{y}_{12} - f_{12}\bar{y}_{21}\right).$$

in which $\left\{\hat{y}_{jq}\right\} \equiv \left[\hat{y}_{12}\ \hat{y}_{21}\ \hat{y}_{22}\right]$. From the definition of $\hat{y}_{11}$, we have

$$\bar{k}_{11} = \frac{\hat{y}_{11} - w_{11}}{1 + r}, \tag{A12}$$

in which w_{11} and r are functions of k_1. Substitute the above equation and the definitions of f_{11} and $\bar{y}_{11}$ into equation (A11), we have

$$\hat{y}_{11}^{1+\rho_1/\eta_{01}}(t) - \hat{f}(t)\hat{y}_{11}^{\rho_1/\eta_{01}} - \bar{f}(t) = 0, \tag{A13}$$

which

$$\hat{f}\left(k_1(t), \left\{\hat{y}_{jq}(t)\right\}\right) \equiv \frac{\Lambda_{11}w_{11}/(1+r) - \left(f_1 + \delta_1 k_1\right)z_1\Lambda_2 - \Lambda_3}{\Lambda_{11}/(1+r) - \left(\xi_1 + \lambda_1\right)\Lambda_2},$$

$$\bar{f}\left(k_1(t), \left\{\hat{y}_{jq}(t)\right\}\right) \equiv \frac{\Lambda_1\bar{\eta}_1\hat{y}_{21}^{\rho_q/\eta_{0q}}}{\Lambda_{11}/(1+r) - \left(\xi_1 + \lambda_1\right)\Lambda_2}.$$

It should be noted that the functions, $\hat{f}$ and $\bar{f}$, are only dependent on the four variables, $k_1(t)$ and $\left\{\hat{y}_{jq}(t)\right\}$. From the definitions of $\left\{\hat{y}_{jq}(t)\right\}$, we see that $\left\{\hat{y}_{jq}(t)\right\}$ are dependent on the four variables, $k_1(t)$ and $\left\{\bar{k}_{jq}(t)\right\}$, where $\left\{\bar{k}_{jq}\right\} \equiv \left[\bar{k}_{12}\ \bar{k}_{21}\ \bar{k}_{22}\right]$. This is important as it will allow us to express the dynamics in a 4-differential equations system. We assume that equation (A13) has a (at least one) positive solution, denoted as

$$\hat{y}_{11} = \Lambda_0\left(k_1, \left\{\bar{k}_{jq}\right\}\right). \tag{A14}$$

This equation is crucial for finding out explicit expressions of the dynamics. In general, equation (A14) may have multiple solutions for any given positive $k_1(t)$ and $\{\bar{k}_{jq}\}$, depending on the parameter value of ρ / η_{01}. By equation (A14), we express $\hat{y}_{11}$ as a function of k_1 and $\{\bar{k}_{jq}\}$. Substitute (A14) into equation (A12)

$$\bar{k}_{11}(t) = \Lambda\left(k_1, \{\bar{k}_{jq}\}\right) \equiv \frac{\Lambda_0\left(k_1, \{\bar{k}_{jq}\}\right) - \phi_{11}(k_1)}{1 - \delta_{k1} + f_1(k_1)}. \tag{A15}$$

Substitute $s_{jq} = \lambda_q \hat{y}_{jq}$ from equations (8) into equations (9)

$$\dot{\bar{k}}_{jq} = \lambda_q \hat{y}_{jq} - \bar{k}_{jq}, \quad j, q = 1, 2.$$

Substitute equations (4), (A14) and (A15) into the above equations

$$\dot{\bar{k}}_{11} = \Lambda_{11}\left(k_1, \{\bar{k}_{jq}\}\right) \equiv \lambda_1 \Lambda_0\left(k_1, \{\bar{k}_{jq}\}\right) - \Lambda\left(k_1, \{\bar{k}_{jq}\}\right), \tag{A16}$$

$$\dot{\bar{k}}_{jq} = \Lambda_{jq}\left(k_1, \{\bar{k}_{jq}\}\right) \equiv \left(\lambda_q f_1(k_1) - \bar{\lambda}_q\right)\bar{k}_j + \lambda_q \phi_{jq}(k_1), \quad (j, q) \in \bar{Q}, \tag{A17}$$

where $\bar{\lambda}_q \equiv 1 - \lambda_q + \lambda_q \delta_{k1}$ and $\bar{Q} \equiv \{(1, 2), (2, 1), (2, 2)\}$.

Taking derivatives of equation (A15) with respect to t yields

$$\dot{\bar{k}}_{11} = \frac{\partial \Lambda}{\partial k_1} \dot{k}_1 + \sum_{(j,q) \in \bar{Q}} \Lambda_{jq} \frac{\partial \Lambda}{\partial \bar{k}_{jq}}, \tag{A18}$$

where we use equations (A17). We don't explicitly calculate partial derivatives as it is straightforward but the expressions are tedious. Equating both right-hand sides of equations (A16) and (A17), we solve

$$\dot{k}_1 = \left[\Lambda_{11}\left(k_1, \{\bar{k}_{jq}\}\right) - \sum_{(j,q) \in \bar{Q}} \Lambda_{jq} \frac{\partial \Lambda}{\partial \bar{k}_{jq}}\right] \left(\frac{\partial \Lambda}{\partial k_1}\right)^{-1}. \tag{A19}$$

We see that $\dot{k}_1$ is a function of $k_1(t)$ and $\{\bar{k}_{jq}(t)\}$. From equations (A15), $\dot{\bar{k}}_{jq}$ are functions of $k_1(t)$ and $\{\bar{k}_{jq}(t)\}$ for $(j, q) = \bar{Q}$. In summary, we obtain Lemma 1.

References

Abel, A.B. and Bernanke, B.S. (1998) Macroeconomics, 3rd edition. New York: Addison-Wesley.

Andersson, A.E., Johansson, B. and Anderson, W.P. (2003, edited) The Economics of Disappearing Distance. Hampshire: Ashgate.

Batten, D.F. (1982) The Interregional Linkages Between National and Regional Input-Output Models. International Regional Science Review 7, 53-67.

Blomquist, G.C., Berger, M.C., and Hoehn, J.C. (1988) New Estimates of Quality of Life in Urban Areas. American Economic Review 78, 89-107.

Brecher, R.A., Chen, Z.Q. and Choudhri, E.U. (2002) Absolute and Comparative Advantage, Reconsidered: The Pattern of International Trade with Optimal Saving. Review of International Economics, 10, 645-656.

Burmeister, E. and Dobell, A.R. (1970) Mathematical Theories of Economic Growth. London: Collier Macmillan Publishers.

Combes, P.G., Duraton, G. and Gobillon, L. (2004) Spatial Wage Disparities: Sorting Matters! CEPR Discussion Paper, 4220.

Deardorff, A.V. (1973) The Gains from Trade in and out of Steady State Growth. Oxford Economic Papers 25, 173-91.

Diamond, D.B. and Tolley, G.S. (1981, Eds.) The Economics of Urban Amenities. New York: Academic Press.

Duranton, G. and Monastiriotis, V. (2002) Mind the Gaps: The Evolution of Regional Earnings Inequalities in the UK 1982-1997. Journal of Regional Science 42, 219-256.

Eaton, J. (1987) A Dynamic Specific-Factors Model of International Trade. Review of Economic Studies 54, 325-338.

Findlay, R. (1984) Growth and Development in Trade Models. In Jones, R.W., Kenen, R.B. (Eds.): Handbook of International Economics. Amsterdam: North-Holland.

Fujita, M. (1989) Urban Economic Theory - Land Use and City Size. Cambridge: Cambridge University Press.

Glaeser, E.L. and Maré, D.C. (2001) Cities and Skills. Journal of Labor Economics 19, 316-42.

Ianchovichina and McDougall, R. (2001) Theoretical Structure of Dynamic GTAP. GTAP Technical Paper, No. 17.

Ikeda, S. and Ono, Y. (1992) Macroeconomic Dynamics in a Multi-Country Economy - A Dynamic Optimization Approach. International Economic Review 33, 629-644.

Kanemoto, Y. (1980) Theories of Urban Externalities. Amsterdam: North-Holland.

McDougall, R. (2002) A New Regional Household Demand System for GTAP. GTAP Technical Paper, No. 20.

Oniki, H. and Uzawa, H. (1965) Patterns of Trade and Investment in a Dynamic Model of International Trade. Review of Economic Studies 32, 15-38.

Roy, J.R. and Johansson, B. (1993) A Model of Trade Flows in Differentiated Goods. The Annals of Regional Science 27, 95-115.

Ruffin, R.J. (1979) Growth and the Long-Run Theory of International Capital Movements. American Economic Review 69, 832-842.

Sorger, G. (2002) On the Multi-Region Version of the Solow-Swan Model. The Japanese Economic Review 54, 146-164.

Takayama, T., Judge, G.G. (1971) Spatial and Temporal Price and Allocation Models. Amsterdam: North-Holland Publishing Company.

Zhang, W.B. (1998) A Two-Region Growth Model - Competition, References, Resources, and Amenities. Papers in Regional Science 77, 173-188.

Zhang, W.B. (2005) Economic Growth Theory. Hampshire: Ashgate.

Zhang, W.B. (2007) A Multi-Region Model with Capital Accumulation and Endogenous Amenities. Environment and Planning A 39, 2248-70.

Zhang, W.B. (2008) International Trade Theory: Capital, Knowledge, Economic Structure, Money and Prices over Time and Space. Berlin: Springer.